

Spectral moments of two-point correlators in perturbation theory and beyond

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Abstract

We discuss the choice of weight functions for the moments of the spectral density for two-point correlators of hadronic currents over a finite energy interval. Of phenomenological relevance is an analysis of the spectra of τ lepton decay on the energy interval $[0, M_\tau]$ and the low-energy hadron production in e^+e^- annihilation. General arguments are given for the calculability of such moments in perturbation theory using both a finite-order analysis and infinite resummation on the contour. Nonperturbative contributions emerging from the operator product expansion for two-point correlators are discussed within explicit models for the physical spectra. The quantitative analysis strongly disfavours weight functions that suppress the high-energy contribution to theoretical moments. This is in agreement with expectations from qualitative considerations in perturbative QCD with asymptotic freedom. We discuss the implication of our results for the ultimate accuracy that can be reached in τ decays and low-energy e^+e^- annihilation into hadrons with present experimental data.

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1 Introduction

The central quantities of interest for studying strong interactions in low-energy hadron phenomenology at accelerators are the spectra of produced particles as being related to the cross section and described by spectral functions. The modern theory of strong interactions (QCD) describes the production processes within perturbative expansions in the strong coupling constant in terms of quarks and gluons and is unable to predict the low-energy hadron spectra pointwise as functions of the energy. Only some integral characteristics of low-energy hadron spectra such as the average over an energy interval can be reliably calculated within perturbative QCD. Theoretically, the procedure of averaging is best justified for observables related to two-point correlators of hadronic currents with well established and simple analytic properties as functions of the energy. Phenomenologically, this is the case for e^+e^- annihilation into hadrons and hadronic τ lepton decays where experimental data of high precision are available.

The process of e^+e^- annihilation into hadrons has been a laboratory for studying strong interactions since long ago where important results have been obtained (as a recent review, see e.g. Ref. [1]). As a famous example of such results one can mention the constraint on the number of colours N_c in QCD obtained from the normalized decay rate $R_{e^+e^-}(s)$ of e^+e^- annihilation into hadrons smoothed over a finite interval at low energies of about $\sqrt{s} \approx 2 \div 3 \text{ GeV}$

$$N_c = \frac{3}{2} R_{e^+e^-}(s) (1 + O(\alpha_s)) \approx 3$$

where α_s is a strong coupling constant. During the last decade much attention has also been paid to a study of hadronic (semileptonic) τ lepton decays which provides a wealth of information on low-energy hadronic physics where the accuracy of experimental data is permanently improving [2, 3]. This makes τ lepton physics an important area of high-precision particle phenomenology.

In the present paper we discuss in some detail the choice of the moments which are given by integrals of the spectral density over the energy range available in experiments. These integrals may contain specific weight functions of different shape that determine the particular form of averaging and the kind of moments. The moments with suitable weight functions can be computed theoretically in perturbative QCD while the detailed pointwise description

of low-energy spectral functions themselves as functions of the energy requires a quantitative understanding of the hadronization mechanism which is lacking at present, especially for light hadrons. The procedure of averaging, in its different modifications, is referred to as a property of duality between hadronic and quark-gluon spectra [4].

While generating a particular set of the moments of the spectral density as observables to confronting theory with experiment, one faces the usual conflict between the precision requirements dictated by theory and experiment. In perturbative QCD a reliable computation is possible only at large energies where the perturbation series converges in a sense that a finite number of high-order corrections in the strong coupling constant (usually one or two) is considered to be numerically small because of the property of asymptotic freedom, so that the moments which emphasize the high-energy part of the spectrum are most reliably calculated. However, such moments are not welcome from the experimental point of view because contributions coming from the high-energy part of the spectrum in the range $0 \leq s \leq M_\tau^2$ usually have got a poor experimental accuracy. On the other hand, moments which suppress the high-energy part of the spectrum are saturated by low-lying resonances or few-particle states which are easier to measure experimentally compared to the states with many particles. This gives rise to better experimental precision for such moments, which would induce on using these moments for a theoretical analysis. However, the theoretical uncertainty for such moments is larger because of both the poor convergence of the perturbation series due to the large value of the effective coupling and the large nonperturbative contributions stemming from the infrared region which are difficult to estimate quantitatively.

The semiphenomenological way for estimating the nonperturbative contributions based on the operator product expansion (OPE) for two-point correlators is most advanced where the knowledge of the numerical values for a few low-dimension terms (gluon condensate and four-quark condensates) fixed from the data. Higher order terms of dimensionality larger than six are virtually unknown; the numerical values that are commonly used derive from some approximations (mainly the factorization into condensates of lower dimensionality) or models (instantons for gluonic condensates) with theoretical uncertainties being scattered over a wide range with rather hypothetical justification in the literature. This makes the estimate of contributions of

nonperturbative terms of higher orders rather imprecise.

Nowadays it is a general trend to consider restrictions imposed by perturbation theory convergence to become more important than those coming from nonperturbative terms. With the knowledge of higher and higher order terms of the perturbation series the asymptotic character of perturbation theory is more and more pronounced which forces one to move the renormalization point to larger and larger energies. In this region nonperturbative terms die out fast because being power corrections they decrease faster than the terms of the logarithmic perturbation series. On the other hand, power corrections reveal the general structure of observables computed in QCD, i.e. they account for nonperturbative effects. In this circumstances there is a tendency to introduce power corrections almost by brute force in case where OPE is not directly applicable because their existence seems to be justified by general principles. Even though the motivation for such a development is clear, the present state of the analysis is definitely far from being quantitative. The obstacle is also that this kind of analysis is infrared dependent, i.e. it depends on the properties of strong interactions at small energies where perturbation theory – the only universal tool of quantitative investigation – is not applicable. This fact makes conclusions based on power corrections depend on the particular model or convention (for instance on a recipe of resummation).

The main purpose of this paper is to give explicit arguments and quantitative estimates for the calculability of the moments of two-point correlators in perturbative QCD and to find a compromise choice of weight functions for the moments that optimizes the precision comparison of theoretical input with experimental data. To be specific, we are talking about the precision analysis of τ lepton decays having been of much interest recently where experimental data are very good. Therefore, we concentrate on the aspects of accuracy of theoretical calculations for the QCD part of the differential τ lepton decay rate and its moments. It is the high precision achieved in the experimental analysis of τ decays and the rather advanced stage of theoretical description that calls for a critical examination of the theoretical formulae used in the analysis of the physics of the τ system. In particular the criteria of reliability of theoretical formulae should be considered carefully. The quantitative discussion of this issue is possible because of the simplicity of the Green's functions necessary to describe the process – only two-point

correlators depending on one single energy variable are relevant. The analytic properties of the two-point correlators in the energy variable are strictly fixed from general principles of quantum field theory. An additional simplification for the renormalization group analysis is provided by the fact that only one scale is involved in the theoretical description.

Theoretical calculations for the τ system can be done within OPE for two-point correlators which contain perturbative and power corrections. Perturbation theory can then be further improved using methods of summation of an infinite number of terms within the renormalization group. We consider these steps in turn. In Sec. 2 we give the basic ideas and fix the notation. In Sec. 3 we consider finite-order perturbation theory, in Sec. 4 we consider power corrections. As an important phenomenological example we consider some peculiarities related to quark mass corrections to current correlators which are important for the determination of the strange quark mass (Sec. 5). In Sec. 6 we consider the infinite resummation of perturbative terms generated by the running of the coupling constant within a renormalization group approach. Sec. 7 contains our conclusions.

2 Setting the stage: details of the theoretical description

Having described the motivation of our work which is rather general and valid for two-point correlators of hadronic currents, we now give the specific details for the description of τ decay observables. Similarities to the e^+e^- annihilation into hadrons are obvious and applications to other hadronic channels are straightforward. In this section we mainly fix our notation; the consideration is rather standard and can be found in the numerous literature on this subject [5, 6].

The semileptonic (hadronic) τ lepton decay is mediated by the charged weak hadronic current of the form

$$j_\mu^w(x) = V_{ud}\bar{u}\gamma_\mu(1 - \gamma_5)d + V_{us}\bar{u}\gamma_\mu(1 - \gamma_5)s \quad (1)$$

where V_{ud} and V_{us} are Cabibbo-Kobayashi-Maskawa matrix elements (elements of the weak mixing matrix). The correlator for the weak hadronic currents in Eq. (1) has the general form

$$\Pi_{\mu\nu}(q^2) = 12\pi^2 i \int \langle T j_\mu(x) j_\nu^\dagger(0) \rangle e^{iqx} dx = q_\mu q_\nu \Pi_q(q^2) + g_{\mu\nu} \Pi_g(q^2) \quad (2)$$

where $\Pi_q(q^2)$ and $\Pi_g(q^2)$ are invariant scalar functions. These scalar functions are further specified depending on which current is considered. In case of the $(\bar{u}d)$ quark current $j_\mu(x) = \bar{u}\gamma_\mu(1 - \gamma_5)d$ (denoted as light quark case) the massless limit is assumed in which case the correlator is transverse, i.e. both invariant functions $\Pi_{q,g}(q^2)$ are expressible through a single scalar correlator function $\Pi_{ud}(q^2)$

$$\Pi_q(q^2) = \Pi_{ud}(q^2), \quad \Pi_g(q^2) = -q^2\Pi_{ud}(q^2). \quad (3)$$

The correlator in case of the $(\bar{u}s)$ quark current (the term proportional to V_{us} , also referred to as the strange quark case) is slightly different as the nonvanishing strange quark mass is taken into account. The relevant formulae are given e.g. in Refs. [6, 7, 8]. We will discuss the strange case later. The rest of the consideration is rather similar to the light quark case. We skip the specification $(\bar{u}d)$ in the following and use $\Pi_{ud}(q^2) \equiv \Pi(q^2)$. We mention that we work within QCD with three light quarks and do not consider corrections due to heavy quarks (c quark) which would enter the calculation at high orders of perturbation theory through loop effects [9, 10, 11]. In addition to the different current specifications as in Eqs. (1) and (3) it is convenient to consider the vector and axial parts of the correlator separately,

$$\Pi_{V+A}(q^2) = \Pi_V(q^2) + \Pi_A(q^2) \quad (4)$$

with $\Pi_V(q^2)$ being related to the pure vector part and $\Pi_A(q^2)$ related to the pure axial part. We introduce the spectral density of a correlator as the discontinuity across the physical cut,

$$\rho(s) = \frac{1}{2\pi i} \text{Disc } \Pi(s) = \frac{1}{2\pi i} (\Pi(s + i0) - \Pi(s - i0)), \quad s > 0 \quad (5)$$

which splits into vector and axial-vector parts accordingly,

$$\rho_{V+A}(s) = \rho_V(s) + \rho_A(s). \quad (6)$$

Here we mainly concentrate on the massless limit for the correlator. The case of the strange quark with nonvanishing mass will be discussed later. In the following the indices V and A will be omitted in the generic case as well if no confusion arises. The correlator in Eq. (2) is normalized to the number of colours N_c (the same holds true for the spectral density which means that $\rho(s) \rightarrow N_c$ for $s \rightarrow \infty$) in the leading parton model approximation with massless

quarks. In the following we will occasionally use also a slightly different normalization which explicitly accounts for the number of colours, resulting in a correlator which is normalized to unity.

The above considerations are general and are also used by experimentalists to classify the appropriate channels: strange particles (K mesons) form the strange channel, non-strange axial-vector mesons (as a_1 and the like) form the axial-vector channel, and the classical vector meson ρ represents the non-strange vector channel. In this respect the pion is somewhat special. It is a Goldstone boson with spin zero and gives a contribution to the axial correlator $\Pi_A(q^2)$ in the massless limit.

Before specifying the theoretical calculations we give the general form of an important observable of τ decays. The basic observable is the normalized τ lepton decay rate for the decay of the τ lepton into hadrons written in the standard form as

$$R_\tau = \frac{\Gamma(\tau \rightarrow \text{hadrons} + \nu_\tau)}{\Gamma(\tau \rightarrow l + \bar{\nu}_l + \nu_\tau)} = N_c S_{\text{EW}} \left(|V_{ud}|^2 (1 + \delta_{ud}) + |V_{us}|^2 (1 + \delta_{us}) \right). \quad (7)$$

The leading terms in Eq. (7) are the parton model results while the terms δ_{ud} and δ_{us} represent the effects of QCD interactions and mass effects (in case of nonvanishing quark masses) [12, 13, 14, 15, 16]. V_{ud} and V_{us} are matrix elements of the weak mixing matrix as defined in Eq. (1), and S_{EW} describes the electroweak radiative corrections to the τ decay rate [17]. Now we take a deeper look into the physics of hadronic τ decays. In the massless limit the expression for the τ decay rate relevant for the QCD part of the quark current is given by the phase-space integral

$$R_\tau^{\text{QCD}} = N_c \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^2 \left(1 + 2 \frac{s}{M_\tau^2} \right) \rho_{V+A}(s) \frac{ds}{M_\tau^2}. \quad (8)$$

The spectral density $\rho(s)$ is related to Adler's function $D(Q^2)$ through the dispersion relation

$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = Q^2 \int \frac{\rho(s) ds}{(s + Q^2)^2} \quad (9)$$

where $Q^2 = -q^2$. At this point theory enters. It is Adler's function $D(Q^2)$ that is most convenient to compute theoretically in the Euclidean domain. It allows one to theoretically predict τ observables through a theoretically computable spectral density $\rho(s)$. Still, to extract the theoretical prediction for $\rho(s)$ from $D(Q^2)$ is not straightforward. The point is that $D(Q^2)$ is only known as an asymptotic expansion at large Euclidean values Q^2 while $\rho(s)$ is obtained

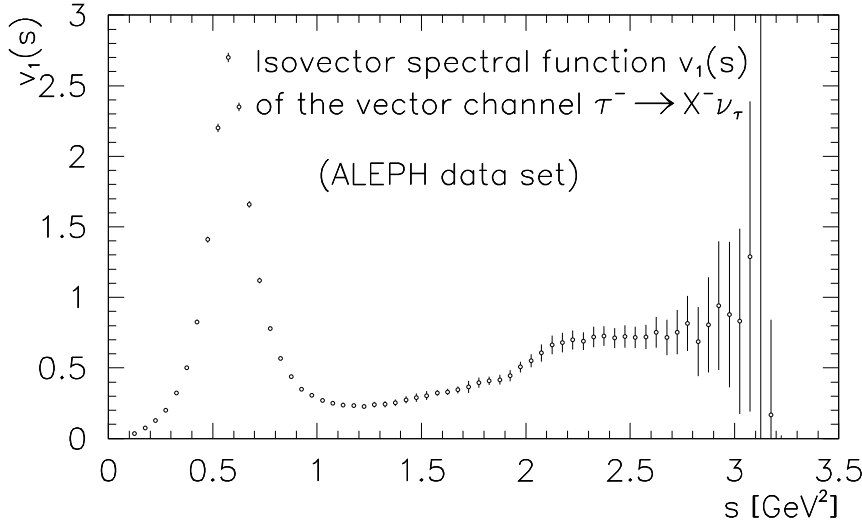


Figure 1: experimentally measured values for the relative cross section in the e^+e^- channel, the data set is taken from Ref. [18].

as a discontinuity of the correlator across the physical cut. Therefore, an analytic continuation into the complex q^2 -plane is necessary. Analytic continuation is an improperly posed problem, i.e. small errors in the initial function $D(Q^2)$ can produce big errors in $\rho(s)$. This instability is especially important for a theoretical computation of $\rho(s)$ at low energies. This situation can also be reformulated in the language of integral equations. The dispersion relation in Eq. (9) gives the spectral density $\rho(s)$ through Adler's function $D(Q^2)$ as a solution of the integral equation. The integral equation given by Eq. (9) is a Fredholm integral equation of the second kind which is known to lead to an improperly posed problem. Thus, errors for the function $\rho(s)$ (as solution of this integral equation) are not continuously related to errors for $D(Q^2)$ (as initial data of the integral equation) and can be very large. The general procedure of constructing approximate solutions for such a problem was suggested by Tikhonov and is known as smoothing. Averaging the spectral density over a finite energy interval (as in sum rules that correspond to the duality concept) can be considered as a particular realization of Tikhonov's smoothing procedure.

The function $\rho(s)$ at low energies is the main object entering Eq. (8), therefore we want to concentrate our studies on it. The counterpart of $\rho(s)$ on the experimental side is the hadronic spectral density $\rho^{\text{had}}(s)$ which can be measured in τ decays in the finite squared energy interval $[0, M_\tau^2]$ with $M_\tau = 1.777 \text{ GeV}$ [3]. The hadronic spectral density $\rho^{\text{had}}(s)$ is a rapidly varying

function in the vicinity of resonances (for the experimental data set in the vector channel see e.g. Fig. 1), therefore there is no hope that one is able to compute this function theoretically pointwise within perturbation theory. This fact is related to the so-called confinement (or hadronization) problem. Instead of a pointwise description of the spectrum at low energies, the appropriate quantities to be analyzed theoretically in perturbative QCD are the moments or integrals of the spectrum with a complete set of weight functions.

We define moments of the spectral density by

$$M_l = (l+1) \int_0^{M_\tau^2} \rho(s) \frac{s^l ds}{(M_\tau^2)^{l+1}} =: 1 + m_l. \quad (10)$$

The factor $(l+1)$ in the definition of the moments is chosen to have all contribution of the parton part uniformly normalized to unity, as it is written explicitly in Eq. (10). This corresponds to the parton contribution which is independent of QCD interactions. Equivalently one can say that all measures

$$(l+1) \frac{s^l}{(M_\tau^2)^{l+1}} ds = d \left(\frac{s}{M_\tau^2} \right)^{l+1} \quad (11)$$

defined on the interval $[0, M_\tau^2]$ are normalized to 1 for the volume of the integration space which in this case is the interval $[0, M_\tau^2]$. The quantities m_l defined in Eq. (10) are related to the interaction. They are of real interest because they contain a wealth of information about the hadronic spectral density $\rho^{\text{had}}(s)$ and are sufficient for theoretical studies.

Theoretical calculations of the moments can be done within OPE which contain perturbative and power corrections. The perturbation theory can then be further improved using methods of summation. We consider these steps in turn.

3 Finite-order perturbation theory

In this section we start the analysis of the moments given in Eq. (10) with the consideration of the ordinary perturbative part of the theoretical spectrum or Adler's function $D(Q^2)$. The theoretical prediction for the function $D(Q^2)$ has been calculated with a very high degree of accuracy within perturbation theory (see e.g. Refs. [19, 20, 21]).

In the massless limit for light quarks (u, d) the perturbation theory expression for Adler's

function is the same for both the axial and the vector channel. In the $\overline{\text{MS}}$ -scheme it reads

$$D(Q^2) = 1 + \frac{\alpha_s}{\pi} + k_1 \left(\frac{\alpha_s}{\pi}\right)^2 + k_2 \left(\frac{\alpha_s}{\pi}\right)^3 + k_3 \left(\frac{\alpha_s}{\pi}\right)^4 + O(\alpha_s^5) \quad (12)$$

where the running coupling α_s is normalized at the scale $\mu^2 = Q^2$, $\alpha_s \equiv \alpha_s(Q^2)$. Here

$$\begin{aligned} k_1 &= \frac{299}{24} - 9\zeta(3) \approx 1.63982, \\ k_2 &= \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5) \approx 6.37101 \end{aligned} \quad (13)$$

for $n_f = 3$ light flavours. $\zeta(z)$ is Riemann's ζ function. The fourth-order $\overline{\text{MS}}$ -scheme coefficient k_3 in Eq. (12) is related to the divergent parts of five-loop diagrams and is not yet known at present. Eqs. (12) and (13) constitute the set of theoretical information necessary for a perturbation theory analysis of the τ system. The analysis of finite-order perturbation theory was presented in Ref. [22].

Starting with Adler's function given in Eq. (12) we deduce the spectral density $\rho(s)$ within the $\overline{\text{MS}}$ -scheme,

$$\rho(s) = 1 + \frac{\alpha_s(s)}{\pi} + k_1 \left(\frac{\alpha_s(s)}{\pi}\right)^2 + \left(k_2 - \frac{\pi^2}{3}\beta_0^2\right) \left(\frac{\alpha_s(s)}{\pi}\right)^3 + \dots \quad (14)$$

where the term proportional to π^2 is a result of the analytic continuation from the Euclidean domain, β_0 is the leading coefficient of the β -function.

The first step in studying the perturbative part of the moments m_l in Eq. (10) is to fix our criteria on how to decide which weight functions are theoretically preferable. In ordinary (or finite order) perturbation theory a natural criterion is the explicit convergence of the perturbation series, i.e. the pattern of numerical behaviour of consecutive terms of the perturbation series. In general, one expects the convergence of the perturbation series to be better for moments which are more 'perturbative', i.e. sensitive to larger scales. However, the explicit convergence can be concealed by the use of a particular scheme, for instance by defining the coupling constant in the $\overline{\text{MS}}$ -scheme which is most widely used. Therefore, the criterion of explicit convergence is scheme dependent. Besides physical reasons of the applicability of perturbation theory to a given observable the explicit convergence for the moments (numerical structure of the perturbation theory series with finite number of terms) can look better or worse depending on the

choice of the expansion parameter in a given scheme. This is an artefact of the definition of the coupling (renormalization or subtraction).

To eliminate such an artificial influence of the scheme definition on physical conclusions one should use scheme independent criteria of theoretical calculability within perturbation theory. The invariant content of the investigation of the spectrum in perturbative QCD, independent of any definition of the charge or the coupling parameter, is the simultaneous analysis of a set of moments. Technically, an efficient way to do this is to factor out all unrelated constants by redefining the charge and introducing an effective coupling which absorbs all constant terms of the perturbative expansion for the polarization function or the spectral density. The purpose of the effective coupling is to get rid of artificial scheme-dependent constants in the perturbation theory expressions. Therefore, in order to analyze a set of moments m_l in the most transparent way (see e.g. Refs. [23, 24, 25]) we express the spectral density in terms of an effective coupling for the τ system.

Note that the use of an effective coupling is a technical trick which simplifies the analysis. Such a parameter is necessary in order to be able to assess the explicit convergence of the perturbation series. The really invariant measure is given by the mutual relations between moments while the effective coupling is still a parameter of the theory to which one cannot prescribe any special physical meaning. Despite this fact, the introduction of a natural internal coupling parameter allows one to extend the perturbation theory series available for the description of the relations between observables by one more term as compared to the analysis in e.g. the $\overline{\text{MS}}$ -scheme (see e.g. Refs. [23, 26]).

In this section we work within finite-order perturbation theory (FOPT) and define an effective coupling $a_M(s)$ on the physical cut for sufficiently large values of s by the relation

$$a_M(s) = \frac{\alpha_s(s)}{\pi} + k_1 \left(\frac{\alpha_s(s)}{\pi} \right)^2 + \left(k_2 - \frac{\pi^2}{3} \beta_0^2 \right) \left(\frac{\alpha_s(s)}{\pi} \right)^3 + \dots \quad (15)$$

such that

$$\rho(s) = 1 + a_M(s). \quad (16)$$

The subscript “ M ” stands for a Minkowskian definition of the effective coupling, i.e. the definition on the physical cut. The decomposition of the spectral density in Eq. (16) reflects the fact that within perturbation theory the correlator contains the parton part which is independent

of α_s . All the constants that may appear in the perturbation theory expression for the spectral density $\rho(s)$ due to a particular choice of the renormalization scheme are absorbed into the definition of the effective charge (see e.g. Refs. [27, 28, 29, 30]), so that only effects of the running of the coupling itself are left. The solution of the evolution equation for the effective coupling,

$$s \frac{da_M(s)}{ds} = \beta(a_M(s)) = -a_M(s) \left(\beta_0 a_M(s) + \beta_1 a_M(s)^2 + \beta_2 a_M(s)^3 + O(a_M(s)^4) \right) \quad (17)$$

resulting from the renormalization group analysis of the correlator with a given effective β -function $\beta(a)$ can be obtained by quadrature,

$$\begin{aligned} a_M(s) &= a_M + \int_{M_\tau^2}^s \beta(a_M(s')) \frac{ds'}{s'} \\ &= a_M + \beta_0 L a_M^2 + (\beta_1 L + \beta_0^2 L^2) a_M^3 + (\beta_2 L + \frac{5}{2} \beta_1 \beta_0 L^2 + \beta_0^3 L^3) a_M^4 + O(a_M^5) \end{aligned} \quad (18)$$

where $a_M = a_M(M_\tau^2)$ and $L = \ln(M_\tau^2/s)$. Moments of the spectral density with different weight functions emphasize different regions of the evolution trajectory of the renormalization group equation. Because we are interested in the running of the effective coupling, we can also use this solution in terms of an expansion in L which is more relevant for our particular setup, instead of an expansion in a_M (which is standard). At any given order of perturbation theory for the effective or for the exact function $\beta(a)$ the running coupling $a_M(s)$ is given by the evolution translation of the initial value a_M according to the renormalization group equation. The result reads

$$\begin{aligned} a_M(s) &= a_M - \beta(a_M) L + \frac{1}{2} \beta(a_M) \frac{\partial \beta(a)}{\partial a} \Big|_{a=a_M} L^2 - \frac{1}{6} \beta(a_M) \frac{\partial}{\partial a} \beta(a_M) \frac{\partial \beta(a)}{\partial a} \Big|_{a=a_M} L^3 + O(L^4) \\ &= \exp \left(-L \beta(a) \frac{\partial}{\partial a} \right) a \Big|_{a=a_M} \end{aligned} \quad (19)$$

where the solution of Eq. (17) is written in a symbolic operator form. Note that this expression is useful for the so-called 't Hooft scheme where, by definition, the β -function of perturbation theory is given by the concise expression $\beta(\alpha) = -\beta_0(\alpha/\pi)^2 - \beta_1(\alpha/\pi)^3$ [31].

Defining the effective coupling $a_M(s)$ directly through the spectrum $\rho(s)$ itself one obtains perturbative corrections to the moments in Eq. (10) only because of running. Without running (as, for instance, in the conformal limit of QCD with a vanishing β -function or at the infrared

fixed point) one would have

$$M_l = 1 + a_M(M_\tau^2) \quad \text{or} \quad m_l = a_M(M_\tau^2) \quad (20)$$

with m_l defined in Eq. (10). In this situation the whole theoretical description of the physics of the τ system in the massless approximation (i.e. without strange particles and including only perturbative corrections without possible power corrections) would reduce to the determination of a single number $a_M(M_\tau^2)$ and consequently there would be no problems with the convergence of the perturbation theory series. This reminds one of the situation in QED where the coupling is defined through the subtraction on the mass shell and simply gives the cross section in the Thomson limit. But because of the running of $a_M(s)$ (which is important numerically because both $a_M(M_\tau^2)$ and β_0 are large), different observables, i.e. different moments of the spectral density, generate different perturbation series from the original object $\rho(s)$ in Eq. (16) and this difference is now within the reach of experiments. Thus, moments just allow one to study the evolution or β -function of the effective coupling $a_M(s)$. We remind the reader that in this section we discuss only the perturbative part of the theoretical spectrum where power corrections are neglected.

The contributions of powers of logarithms (from Eqs. (18) or (19)) to the normalized moments in Eq. (10) are given by

$$(l+1) \int_0^{M_\tau^2} \ln\left(\frac{M_\tau^2}{s}\right) \frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{1}{l+1} \quad (21)$$

and

$$(l+1) \int_0^{M_\tau^2} \ln^2\left(\frac{M_\tau^2}{s}\right) \frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{2}{(l+1)^2}. \quad (22)$$

A general formula for an arbitrary (integer) power of the logarithm reads

$$(l+1) \int_0^{M_\tau^2} \ln^p\left(\frac{M_\tau^2}{s}\right) \frac{s^l ds}{(M_\tau^2)^{l+1}} = \frac{p!}{(l+1)^p}. \quad (23)$$

Therefore, at any fixed order of the perturbation series expansion the effects of running die out for large values of l improving the (explicit) asymptotic structure of the perturbative series for the moments in Eq. (10). This is obvious and expected in QCD with its property of asymptotic freedom because the moments with weight functions s^l suppress the infrared (small s) region of integration where perturbation theory is not applicable. Note that for a given order of

perturbation theory the real effects of running are connected with powers of logarithms and not with powers of the coupling a_M . Therefore one computes an expansion of the spectral density $\rho(s)$ as a Taylor series with the initial value at M_τ ,

$$\rho(s) = \rho_0(M_\tau) + \rho_1(M_\tau)L + \rho_2(M_\tau)L^2 + \dots \quad (24)$$

It is just the coefficients of this Taylor series (coefficients $\rho_i(M_\tau)$ of the powers of logarithms) that are calculated by a perturbative expansion in the coupling constant. In the effective scheme using a_M one finds

$$\begin{aligned} \rho_0(M_\tau) &= a_M, \\ \rho_1(M_\tau) &= -\beta(a_M), \\ \rho_2(M_\tau) &= \frac{1}{2}\beta(a_M)\frac{\partial\beta(a_M)}{\partial a_M}, \dots \end{aligned} \quad (25)$$

while for the $\overline{\text{MS}}$ -scheme one would have

$$\rho_0(M_\tau) = \frac{\alpha_s(M_\tau)}{\pi} + k_1 \left(\frac{\alpha_s(M_\tau)}{\pi} \right)^2 + \left(k_2 - \frac{\pi^2}{3}\beta_0^2 \right) \left(\frac{\alpha_s(M_\tau)}{\pi} \right)^3 + \dots \quad (26)$$

The technical advantage of introducing the effective scheme becomes obvious. The effects of running as given by powers of logarithms can be obtained in a concise form as explicit functions of a_M from Eq. (23) if the β -function is known. Note that only for the leading order β -function the powers of logarithms coincide with the powers of the coupling a .

The behaviour at fixed l and large p , i.e. the convergence (or asymptotic structure) of the perturbation series with an infinite number of terms for the moments is another story. This question was intensively discussed in the literature. We present our analysis of this problem in Sec. 6.

Therefore, the set of moments in Eq. (10) has a simple and clear structure of perturbative convergence in finite-order perturbation theory: larger values of l are better behaved from the perturbation theory point of view. However, in practice and for an efficient comparison with experimental data the choice of weight functions is also dictated by the precision of the available experimental data. With this constraint taken into account, s^l -moments for large values of l are not welcome from an experimental point of view. They are dominated by the contributions coming from the high-energy part of the τ decay spectrum (and therefore show

a better perturbative convergence) while the experimental accuracy for the moments basically deteriorates with increasing l because poorly known contributions close to the high-energy end of the interval are enhanced. Therefore, with respect to the precision aimed for the the structure of the set of the s^l -moments on the experimental side is opposite to that on the theory side: for larger l the precision for the calculation of moments using experimental data is worsening fast.

To balance the precision requirements from the experimental and theoretical sides one can consider a modification of the set of moments (see e.g. Refs. [32, 33]). Note that in general any polynomial in s can be used as a weight function for a finite interval. In a sense a complete system of orthogonal polynomials (like the Jakopian polynomials $P^{(a,b)}(x)$ with the rather general weight function $x^a(1-x)^b$ which are orthogonal on the finite interval $(0, 1)$ [34]) could provide an exact expansion of the function $\rho(s)$ in term of its moments and allows for its full reconstruction of the special shape of the coefficients of the expansion (supposing that $\rho(s)$ is a continuous function). Therefore, the particular choice of weight functions is determined by the efficiency of solving concrete problems. Actually, the basis $\{s^l; l = 0, \dots, \infty\}$ is complete for continuous functions on the interval $[0, M_\tau^2]$, and the choice of this basis allows for the most direct evaluation of the structure of the moments from the perturbation theory point of view which is the most relevant for the theoretical analysis. At $s = M_\tau^2$ the spectrum is quite perturbative and for large values of l the s^l -moments are basically determined by $\rho(M_\tau^2)$.

As the simplest modification done in order to suppress experimental errors from the high-energy end of the spectrum, we introduce the system of modified moments with weight functions

$$w_{kl}(s) = \frac{(k+l+1)!}{k!l!} \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l. \quad (27)$$

Even in the case $l = 0$ we can consider the modified moments as mixed moments, because they are just linear combinations of the direct moments with weight functions s^l as given in Eq. (10).

The modified moments

$$M_{kl} = \frac{(k+l+1)!}{k!l!} \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{\rho(s)ds}{M_\tau^2} \equiv 1 + m_{kl} \quad (28)$$

are normalized to unity. Within the set given in Eq. (28) the best choice from the experimental point of view is to use large values for k and small values for l . This choice was also advocated

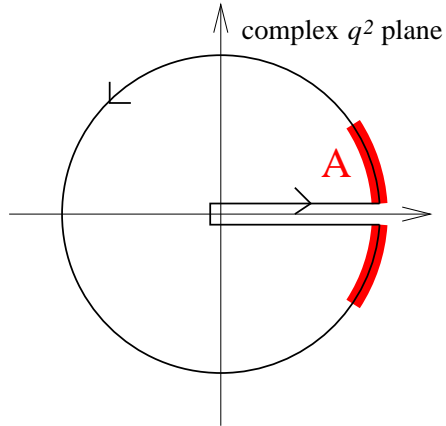


Figure 2: contour of integration in the complex q^2 -plane

to be justified theoretically as improving the precision based on the integration over the contour in the complex q^2 -plane (see Fig. 2). The reasoning was that the weight functions $(1 - s/M_\tau^2)^k$ suppress the contribution of that part of the contour that is close to the real positive semi-axis where OPE is not applicable (region A in Fig. 2) which in turn, according to standard wisdom, can improve the accuracy of theoretical predictions. Below we give arguments that this is not the case: the weight functions $(1 - s/M_\tau^2)^k$ simply ruin the perturbative structure of the moments for large values of k . Let us demonstrate this in more detail.

Within finite-order perturbation theory the integration over the contour [35, 36, 37] is completely equivalent to the integration over the cut along the positive semi-axis because of the analytic properties of the functions $\ln^p(-M_\tau^2/q^2)$ occurring in the expansion of the correlator. This can easily be seen from the spectral density according to Eq. (16). All polynomial moments can be rewritten as contour integrals in the complex q^2 -plane. Therefore, for finite-order perturbation theory it makes no difference which particular representation for the moments is used: integrals over the contour in the complex q^2 -plane and integrals over the cut along the positive semi-axis are mathematically equivalent at any order of perturbation theory. Power corrections and infinite summation in perturbation theory will be discussed later on.

In the framework of finite-order perturbation theory we will discuss in the following our statement that weight functions $(1 - s/M_\tau^2)^k$ with large k are a bad choice for moments from the perturbation theory point of view, i.e. they are strongly nonperturbative. Indeed, the weight function in Eq. (27) has its maximum value at $s_{\max} = M_\tau^2 l / (l + k)$ (see Fig. 3). The integral

in Eq. (28) is dominated by contributions from around this value s_{\max} for a smooth function $\rho(s)$ as it is given by perturbation theory. The disadvantage of choosing such moments is that the factor $(1 - s/M_\tau^2)^k$ strongly enhances the infrared region of integration where perturbation theory is not valid. This can be seen by looking at the explicit convergence of the perturbation series for the moments [7].

To show this explicitly, we consider the integration of the running coupling with the modified weight functions in Eq. (27). The analogues of Eqs. (21), (22), and (23) are necessary in order to find the behaviour of contributions due to logarithms (see Eq. (19)). The values of the coefficients of such contributions can be readily found in a concise form for arbitrary values of k at any giving finite order of perturbation theory (for a given power of the logarithm). For instance, the contribution of the term proportional to $L = \ln(M_\tau^2/s)$ is given by

$$(k+1) \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \ln\left(\frac{M_\tau^2}{s}\right) \frac{ds}{M_\tau^2} = \sum_{j=1}^{k+1} \frac{1}{j} = \gamma_E + \psi(k+2) \quad (29)$$

where γ_E is Euler's constant and $\psi(z)$ is the digamma function. In contrast to Eq. (21) the contributions in Eq. (29) increase as $\ln(k)$ for large k . so the coefficients of the perturbation series for the moments grow relatively larger with increasing k . The contribution of the L^2 -term reads

$$(k+1) \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \ln^2\left(\frac{M_\tau^2}{s}\right) \frac{ds}{M_\tau^2} = \left(\sum_{j=1}^{k+1} \frac{1}{j}\right)^2 + \sum_{j=1}^{k+1} \frac{1}{j^2} = (\gamma_E + \psi(k+2))^2 + \psi'(k+2) \quad (30)$$

where $\psi'(z)$ is the first derivative of the digamma function. The coefficients in Eq. (30) grow as $\ln^2(k)$ for large k which has to be compared with Eq. (22). For higher powers of L the expressions become too lengthy to be presented here. The asymptotic behaviour of the perturbation theory coefficients for large k and fixed p is totally different from the behaviour obtained for the s^l -moments, the direct moments from Eq. (10).

The property of convergence of the resulting perturbation series for the moments can be reformulated in the language of effective scales for the moments themselves. This is similar to the approach where moments are reexpressed by moments [23]. Indeed, let us take the expansion of Eq. (18) up to next-to-leading order,

$$a_M(s) = a_M + \beta_0 L a_M^2 + O(a_M^3) = a_M + \beta_0 a_M^2 \ln(M_\tau^2/s) + O(a_M^3) \quad (31)$$

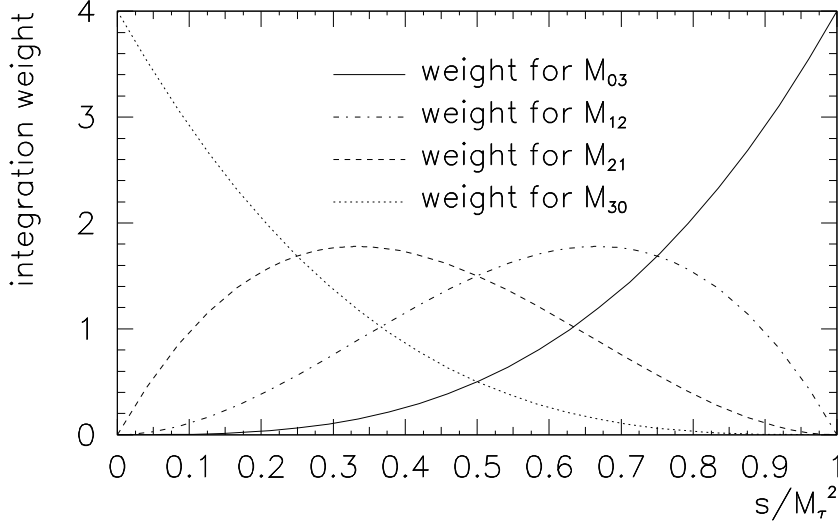


Figure 3: Different weight functions $w_{kl}(s)$ for $(k, l) = (3, 0)$, $(2, 1)$, $(1, 2)$, and $(0, 3)$.

Then for the s^l -moments (i.e. the moments M_{0l} of Eq. (28)) one has

$$m_{0l} = a_M + \frac{\beta_0}{l+1} a_M^2 + O(a_M^3) \quad (32)$$

which translates into the effective scale $M_\tau^2 e^{-1/(l+1)}$ through the relation

$$a_M + \frac{\beta_0 a_M^2}{l+1} + O(a_M^3) = a_M(M_\tau^2 e^{-1/(l+1)}) + O(a_M^3). \quad (33)$$

On the other hand, for the $(M_\tau^2 - s)^k$ -moments given by Eq. (28) with large values for k and $l = 0$ the result of integrating Eq. (31) reads

$$m_{k0} = a_M + \beta_0 a_M^2 \ln(k) + O(a_M^3) = a_M(M_\tau^2/k) + O(a_M^3). \quad (34)$$

This leads to the effective scale M_τ^2/k for large k . The results of these explicit calculations agree with a qualitative estimate based on the observation that the essential region of integration where integrals for the moments are saturated for reasonably smooth functions $\rho(s)$ (which are just powers of logarithms in finite-order perturbation theory) is located around $s_{\max} = M_\tau^2 l/(l+k)$. Obviously, the quantity $a_M(M_\tau^2/k)$ cannot be evaluated in perturbative QCD for large values of k . To be specific, for $k = 2$ we have

$$m_{20} = a_M + \beta_0 a_M^2 \frac{11}{6} + O(a_M^3) = a_M(M_\tau^2 e^{-11/6}) + O(a_M^3) = a_M(0.16 M_\tau^2) + O(a_M^3), \quad (35)$$

and for $k = 3$

$$m_{30} = a_M + \beta_0 a_M^2 \frac{25}{12} + O(a_M^3) = a_M(M_\tau^2 e^{-25/12}) + O(a_M^3) = a_M(0.12 M_\tau^2) + O(a_M^3). \quad (36)$$

Thus we find that ,in the language of effective scales, even a value of $k = 3$ requires the calculation of the effective coupling at a low scale which is definitely located in a nonperturbative regime. In this context we remind the reader that the effective scale Λ_M for the effective coupling a_M as defined in Eq. (15) is given by

$$\Lambda_M^2 = \exp(k_1/\beta_0)\Lambda_{\overline{\text{MS}}}^2 = 2.07\Lambda_{\overline{\text{MS}}}^2. \quad (37)$$

To fix the scale we take the value of $\Lambda_{\overline{\text{MS}}} = 350 \text{ MeV}$ as determined from τ decays [38]. Then the process dependent parameter Λ_M determining the effective coupling is numerically given by

$$\Lambda_M^2 = 2.07 \times (0.35 \text{ GeV})^2 \approx 0.1 \text{ GeV}^2. \quad (38)$$

This is to be compared with the effective scale from Eqs. (35) and (36). For $M_\tau = 1.777 \text{ GeV}$ one finds $0.12 \times M_\tau^2 = 0.36 \text{ GeV}^2$ which is definitely located in the nonperturbative region for the process under consideration.

This is the situation with finite-order perturbation theory which is expected on the basis of asymptotic freedom of QCD and which is readily confirmed by a direct calculation. As the main output of this analysis we observe the movement of the related scale to where integrals are saturated and conclude that $(M_\tau^2 - s)^k$ -moments for large k cannot be calculated in perturbation theory. Because the picture of explicit convergence due to the effects of running can be concealed by the use of a particular scheme (the $\overline{\text{MS}}$ -scheme, for instance) we have discussed the essence of the situation in an effective scheme where only effects of running are seen for the moments but no artificial constants related to the definition of an expansion parameter in a specific scheme occur.

To conclude this section, we summarize our findings saying that demands from the theory side, basically dictated by the calculability in perturbation theory, and from the experiment side, basically dictated by the precision of experimental data, lead to conflicting requirements for the choice of weight functions as for where they should give the dominant contribution. Thus, one faces the usual conflict between experimental and theoretical accuracy which in our case is reflected by the range of (k, l) -values for the modified moments that are chosen as optimal observables. Having explicit perturbation theory formulae at hand (see e.g. Eq. (12), (14), (15), (16), (17), and (18)) one can establish the ultimate theoretical accuracy implied by

the asymptotic character of the perturbation series for a given experimental observable with any stated precision. This allows one to conclude which error – experimental or theoretical – dominates the uncertainty of an observable related to τ decay physics. From the point of view of perturbation theory, large values of l are preferable while large values of k cannot be used.

However, perturbation theory is not the end of the story. Instead, if the convergence is not satisfactory, this is an indicator that something is going wrong. There are contributions beyond perturbation theory, so if perturbation theory convergence is slow, perturbation theory itself is not reliable and nonperturbative terms become important. Nonperturbative terms for two-point correlators can be analyzed within OPE which can (and people believe, does) describe the spectrum at low energies better than ordinary perturbation theory.

4 Power corrections

In this section we consider the contribution of power corrections within OPE to the systems of moments given in Eqs. (10) and (28), and the interplay between the magnitude of this contribution and the structure of the perturbation theory series. In doing so we neglect, as usual, the weak $\ln(Q^2)$ dependence of the coefficient functions of local operators within OPE stemming from the anomalous dimension of the local operators to see the net effect of a power-type behaviour. Though this is a rather common practice in phenomenological applications, we do it for simplicity only; this dependence can be readily taken into account (see e.g. the full analysis of a realistic case below). The standard power corrections due to nonvanishing vacuum expectation values of local operators within OPE are relatively small for τ decays but can play a crucial role for particular observables determined by special weight functions that lead to nonperturbative moments. Indeed, we saw in the previous section that $(M_\tau^2 - s)^k$ -moments are not computable in perturbation theory for large values of k . Here we will see that one needs all power corrections which are known (or at least many terms) to bring the experimental results for the $(M_\tau^2 - s)^k$ -moments in consistency with the theoretical framework based on OPE, i.e. the expansion in power corrections.

For the system of moments in Eq. (10) the contribution of power corrections reduces to a single term of the form $(\Lambda^2/M_\tau^2)^l$ which decreases with l for $\Lambda < M_\tau$ where Λ is a typical scale

of the power corrections related to the nonperturbative scale of QCD. This restriction for Λ is a necessary condition for the applicability of perturbation theory. In fact, for $\Lambda^2/M_\tau^2 < 1$ the perturbation theory contribution dominates in the total result for the direct s^l -moments, being a necessary condition for using the technique. It is not quite clear, however, which numerical value should be chosen for Λ . The choice $\Lambda \sim \Lambda_{\text{QCD}}$ is just too vague. In practice, a better choice for the scale could be $\Lambda \sim m_\rho$ or the mass of the respective low-lying resonance in the corresponding channel as for example the mass of the a_1 meson for the axial correlator (see the discussion below).

The perturbative contribution to the s^l -moments with large values of l is saturated by high-energy contributions and therefore converges perturbatively, the convergence becomes even better with increasing l . Therefore, the s^l -moments are theoretically computable with a strict control over the precision in perturbation theory. On the contrary, for $l \sim 0$ and large values of k the system of mixed moments in Eq. (28) (which we call large k moments in the following) is saturated by low-energy contributions, i.e. basically by the contribution of the ground-state resonance, the existence of which is a common feature of the low-energy part of hadronic spectra. The large k moments are therefore completely nonperturbative. Power corrections to the large k moments come from many terms of OPE with local operators of high dimensionality. While these contributions are certainly decisive for reproducing the experimental spectrum, nothing definite can be said theoretically about such a sum of power corrections in any realistic case because high-dimensional power corrections are completely unknown numerically.

An instructive example for the importance of power corrections to mixed moments with large k needed to reproduce the experimental spectrum of hadrons at low energies is the comparison of moments for vector and axial channels. In the massless limit the perturbation theory series for the spectral density is the same both for the vector and axial channel leading to identical expressions for the large k moments in perturbation theory. Experimentally, the large k moments are saturated by the contributions of the first resonances which are completely different for both channels: the pion as a Goldstone boson in the axial channel and the ρ meson in the vector channel. Therefore, no method of summation of perturbation theory series alone can bring the perturbation theory results for the mixed moments with large k in agreement

with the experiment: if the method of summation is a regular one, the result is the same for both channels because it sums the same initial series. In this case perturbation theory for large k moments is in trouble and the power corrections have to provide the correct result. Because the asymptotic regime of very large k requires the consideration of all power corrections, this example shows that large k moments as in Eq. (28) cannot be used within the perturbation theory framework even if they are preferable from the experimental point of view. On the contrary, the system of direct moments in Eq. (10) (also called l moments) can be calculated reliably in perturbation theory. These moments are saturated by perturbation theory which is also valid phenomenologically and known since long ago (chiral invariance, Weinberg sum rules) while the contribution of power corrections in this case is small and provides a necessary fine-tuning for better accuracy.

Another way to understand what has been said up to now is to state that mixed moments with large values for k resolve the point-by-point structure of the spectral density at the origin which is definitely nonperturbative while mixed moments for large values for l resolve the point-by-point structure of the spectral density at large values for s where perturbation theory is more reliable and the standard asymptotic analysis is possible.

After these preliminary remarks we give a realistic example where the magnitude of power corrections and perturbative terms can be seen quantitatively. In the consideration below we use the vector channel for the e^+e^- annihilation into hadrons as an example which is relevant for the calculation of the electromagnetic coupling constant at M_Z . This is the classical channel for vector mesons (ρ, ω). We give some details to elaborate the real situation in QCD within OPE. This example serves as a base for the next step – to simplify this picture to a model which contains only the gross features of the phenomenological structure of the spectrum but is simple and efficient for the purpose of investigating higher order power corrections within OPE.

4.1 The vector channel contribution

In this subsection we discuss the relative magnitude of perturbative and power corrections for a particular quantity related to the two-point correlator of hadronic currents of light quarks.

The quantity which is chosen is the value of the vacuum polarization function at the origin. This quantity is infrared (IR) sensitive, a fact that allows one to see the interplay between perturbative and power corrections.

For light quarks the perturbative part of the vector (and axial) correlator is calculable for large values of $Q^2 = -q^2$ within the Euclidean domain. For the $\overline{\text{MS}}$ renormalization scheme it is given by

$$\begin{aligned} \Pi^{\text{light}}(\mu^2, Q^2) = & \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{5}{3} + \frac{\alpha_s}{\pi} \left(\ln\left(\frac{\mu^2}{Q^2}\right) + \frac{55}{12} - 4\zeta(3) \right) \\ & + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{9}{8} \ln^2\left(\frac{\mu^2}{Q^2}\right) + \left(\frac{299}{24} - 9\zeta(3)\right) \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5) \right). \end{aligned} \quad (39)$$

Eq. (39) is written for $n_f = 3$ active light quarks with the effective coupling $\alpha_s \equiv \alpha_s^{(3)}(\mu^2)$. One is interested in an estimate for the value $\Pi^{\text{light}}(\mu^2, 0)$. However, $\Pi^{\text{light}}(\mu^2, Q^2 \rightarrow 0)$ cannot be obtained from Eq. (39) because there is no scale for light quarks, one can therefore not avail of a perturbative expression. Because singularities at small momenta are related to IR problems, it suffices to modify only the IR structure of the correlator $\Pi^{\text{light}}(\mu^2, Q^2)$ in order to obtain an expression for all values of Q^2 . It is convenient to modify just the contribution of low-energy states to the correlator by using the dispersion relation

$$\Pi^{\text{light}}(Q^2) = \int_0^\infty \frac{\rho^{\text{light}}(s) ds}{s + Q^2} \quad (40)$$

where dimensional regularization is understood to be used for $\rho^{\text{light}}(s)$. In fact, Eq. (40) can be used for the bare quantities $\Pi^{\text{light}}(Q^2)$ and $\rho^{\text{light}}(s)$. The modification should be done locally (i.e. with a finite support in the squared energy variable s in Eq. (40)) without changing the perturbative behaviour, in order not to affect any ultraviolet (UV) properties (μ^2 dependence) of the correlator $\Pi^{\text{light}}(\mu^2, Q^2)$. This requirement is important for retaining the renormalization group properties of the correlator. The low-energy modification of the perturbative expression for the spectrum is inspired by experiment: at low energies there is a well-pronounced bound state as a result of the strong interaction between quarks. Therefore, we adopt a model of the IR modification where the high-energy tail of the integral in Eq. (40) is computed within perturbation theory retaining the renormalization group structure of the result while in the low-energy domain there is a contribution of a single resonance.

For a generic light quark correlator in the massless perturbative approximation one introduces the IR modification

$$\rho^{\text{light}}(s) \rightarrow \rho_{\text{IRmod}}^{\text{light}}(s) = F_R \delta(s - m_R^2) + \rho^{\text{light}}(s) \theta(s - s_0) \quad (41)$$

where F_R , m_R , and s_0 are IR parameters of the spectrum. Note that they may not correspond to the actual experimental numbers. Substituting the IR modified spectrum in Eq. (41) into Eq. (40), one finds

$$\begin{aligned} \Pi_{\text{IRmod}}^{\text{light}}(\mu^2, 0) &= \frac{F_R}{m_R^2} + \ln\left(\frac{\mu^2}{s_0}\right) + \frac{5}{3} + \frac{\alpha_s}{\pi} \left(\ln\left(\frac{\mu^2}{s_0}\right) + \frac{55}{12} - 4\zeta(3) \right) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{9}{8} \ln^2\left(\frac{\mu^2}{s_0}\right) + \left(\frac{299}{24} - 9\zeta(3)\right) \ln\left(\frac{\mu^2}{s_0}\right) + \frac{34525}{864} - \frac{715}{18}\zeta(3) + \frac{25}{3}\zeta(5) - \frac{3\pi^2}{8} \right). \end{aligned} \quad (42)$$

We identify m_R with a mass of the low-lying resonance which is the only input giving a scale to the problem. The IR modifying parameters F_R and s_0 are fixed from quark-hadron duality arguments. Notice the $O(\alpha_s^2)$ difference between Eq. (39) and Eq. (42) given by the additional term $-3\pi^2/8$. This is the so-called “ π^2 -correction” (see e.g. Ref. [39]) which can be rewritten through $\zeta(2) = \pi^2/6$.

Next we use the OPE with power corrections that semi-phenomenologically encode the information about the low-energy domain of the spectrum through the vacuum condensates of local gauge invariant operators [40] in order to supplement the perturbative contribution. The OPE for the light quark correlator reads

$$\Pi_{\text{OPE}}^{\text{light}}(\mu^2, Q^2) = \Pi_{\text{IRmod}}^{\text{light}}(\mu^2, Q^2) + \frac{\langle \mathcal{O}_4 \rangle}{Q^4} + \frac{\langle \mathcal{O}_6 \rangle}{Q^6} + \dots \quad (43)$$

The quantities $\langle \mathcal{O}_{4,6} \rangle$ denote nonperturbative contributions of dimension-four and dimension-six vacuum condensates. These contributions are UV soft (they do not change the short distance properties of the correlator) and are related to the IR modification of the spectrum. For the purposes of fixing the numerical values of the parameters F_R and s_0 which describe the IR modification of the spectrum one needs only the first two power corrections $1/Q^2$ and $1/Q^4$. Note, however, that the coefficient of the $1/Q^2$ correction vanishes because there are no gauge invariant dimension-two operators in the massless limit. Computing the IR modified and OPE supplemented correlation function, we find finite energy sum rules (FESR) that allow one to fix

the parameters F_R and s_0 [41],

$$\begin{aligned} F_R &= s_0 \left\{ 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi} \right)^2 \left(\beta_0 \ln \left(\frac{\mu^2}{s_0} \right) + k_1 + \beta_0 \right) \right\} + O(\alpha_s^3), \\ F_R m_R^2 &= \frac{s_0^2}{2} \left\{ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi} \left(\beta_0 \ln \left(\frac{\mu^2}{s_0} \right) + k_1 + \frac{\beta_0}{2} \right) \right\} - \langle \mathcal{O}_4 \rangle + O(\alpha_s^3). \end{aligned} \quad (44)$$

We treat $\langle \mathcal{O}_4 \rangle$ as a small correction and take its coefficient function as a constant (the total contribution is renormalization group invariant). Eqs. (44) fix F_R and s_0 through m_R^2 and $\langle \mathcal{O}_4 \rangle$. In using higher order terms in the OPE expansion (for instance, $\langle \mathcal{O}_6 \rangle / Q^6$) one could avoid substituting m_R^2 from experiment because within the IR modification given in Eq. (41) the IR scale is determined by the dimension-six vacuum condensate $\langle \mathcal{O}_6 \rangle$ [41]. We do not do this because the primary purpose of the present analysis is to find the low-scale normalization for the electromagnetic coupling and not to describe the spectrum in the low-energy domain. The use of the experimental value for the resonance mass m_R^2 makes the calculation more precise because the numerical value for the $\langle \mathcal{O}_6 \rangle$ condensate is not known well (cf. Ref. [42]).

The leading order solution for Eqs. (44) (upon neglecting the perturbative and nonperturbative corrections) is given by the parton model result $s_0 = 2m_R^2$ and $F_R = s_0 = 2m_R^2$ which is rather precise. This solution has been used for predicting masses and residues of the radial excitations of vector mesons within the local duality approach where the experimental spectrum is approximated by a sequence of infinitely narrow resonances [43]. The condensate of dimension-four operators for light quarks is given by

$$\langle \mathcal{O}_4 \rangle = \frac{\pi^2}{3} \left(1 + \frac{7\alpha_s}{6\pi} \right) \langle \frac{\alpha_s}{\pi} G^2 \rangle + 2\pi^2 \left(1 + \frac{\alpha_s}{3\pi} \right) (m_u + m_d) (\langle \bar{u}u \rangle + \langle \bar{d}d \rangle). \quad (45)$$

The order α_s corrections to the gluon condensate were calculated in Ref. [44]. We retain small corrections proportional to the light quark masses and treat them according to the approximation of isotopic symmetry for the light quark condensates, $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$, which is rather precise for u and d quarks. The quark condensate part of Eq. (45) is given by the partially conserved axial current (PCAC) relation for the π meson,

$$(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle = -f_\pi^2 m_\pi^2. \quad (46)$$

Here $f_\pi = 133 \text{ MeV}$ is the decay constant of the charged pion and $m_\pi = 139.6 \text{ MeV}$ is the mass of the charged pion. For the numerical standard value $\langle (\alpha_s/\pi) G^2 \rangle = 0.012 \text{ GeV}^4$ of the gluon

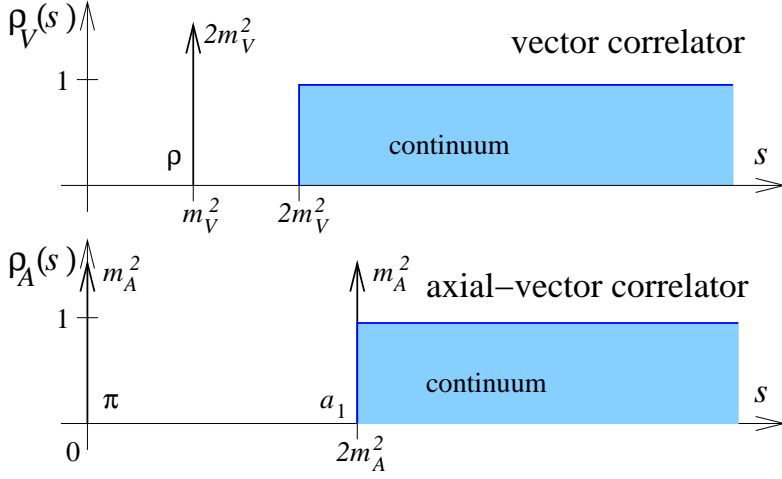


Figure 4: Spectral densities for the models for the vector channel (corresponding to Eq. (48)) and the axial-vector channel (corresponding to Eq. (57)). The arrows indicate the narrow width resonances, given by distributions.

condensate [40] and $\alpha_s/\pi = 0.1$ one finds

$$\langle \mathcal{O}_4 \rangle = \frac{\pi^2}{3} \left(1 + \frac{7\alpha_s}{6\pi} \right) \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle - 2\pi^2 \left(1 + \frac{\alpha_s}{3\pi} \right) f_\pi^2 m_\pi^2 = 0.037 \text{ GeV}^4. \quad (47)$$

The relation $s_0 = 2m_\rho^2$, where $m_\rho = 768.5 \text{ MeV}$ is a mass of the lowest (ρ meson) resonance in the non-strange isotopic $I = 1$ vector channel, is rather precise numerically. The gluon condensate gives a small correction to the basic duality relation $s_0 = 2m_R^2$ for light quarks. The same is true for the $I = 0$ channel where the lowest resonance is the ω meson with a mass $m_\omega = 781.94 \text{ MeV}$. So we conclude that the model in Eq. (41) is rather accurate if the scale is fixed by the low-lying resonance.

4.2 Models for the vector and axial-vector channel

Based on the above example we investigate models for non-perturbative effects resulting from standard power corrections [40] that allow us to obtain all power corrections in a concise form. We adhere to models for the vector and axial channels which are simple enough to be calculable and which retain the main features of the spectra. These models are depicted in Fig. 4. With these models at hand we have all power corrections available for the discussion of their role in moments for the spectral density in different channels.

As a model for the vector channel we take the spectral density of the previous subsection in a simplified form ($m \equiv m_V$),

$$\rho_V(s) = 2m^2\delta(s - m^2) + \theta(s - 2m^2). \quad (48)$$

By using the dispersion relation we obtain the correlation function

$$\Pi_V(Q^2) = \frac{2m^2}{m^2 + Q^2} + \ln\left(\frac{\mu^2}{2m^2 + Q^2}\right) + \text{subtractions} \quad (49)$$

with necessary subtractions. The expression in Eq. (49) is used to generate all power corrections. Indeed, for $Q^2 \gg m^2$ one finds

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \sum_{n=1}^{\infty} \left(-\frac{2m^2}{Q^2}\right)^n \left(\frac{1}{n} - \frac{1}{2^{n-1}}\right) \quad (50)$$

where the first term is the leading order perturbative contribution in α_s and the remaining terms are power corrections. Note that the analytic properties of the expansion in Eq. (50) up to any finite order are different from the exact (though model-dependent) result in Eq. (49). This is rather a common feature: analytic properties of approximations for the correlators can be different from those of the exact result. In some instances this restricts the precision and may lead to a misuse of approximations in areas where they do not work.

Writing the OPE for the correlator in the general form

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \sum_{n=1}^{\infty} \frac{c_n}{(Q^2)^n}, \quad (51)$$

for the model in Eq. (49) we find

$$c_n = \left(\frac{1}{n} - \frac{1}{2^{n-1}}\right) (-2m^2)^n. \quad (52)$$

The first two coefficients vanish, $c_1 = c_2 = 0$. The vanishing of c_1 is in full agreement with the fact that there exist no dimension-two operators in realistic cases while the vanishing of c_2 means that the gluon condensate is neglected in this model (which is justified numerically for the realistic case of τ decays, see also the previous section). Note that these two constraints are built-in requirements for our models. We just constructed the models in this way. The third coefficient reads

$$c_3 = -\frac{2}{3}m^6. \quad (53)$$

Phenomenologically this coefficient is related to the vacuum expectation value of local four-quark operators which in factorized approximation is given by

$$\langle \mathcal{O}_6^V \rangle = -\frac{896\pi^3}{81}\alpha_s \langle \bar{q}q \rangle^2. \quad (54)$$

Its numerical value is approximated reasonably well by the expression in Eq. (53) with $m = m_\rho$. For further comparison we give the value of the dimension-six term in the axial channel,

$$\langle \mathcal{O}_6^A \rangle = \frac{1408\pi^3}{81}\alpha_s \langle \bar{q}q \rangle^2. \quad (55)$$

This vacuum expectation value is larger and has a sign opposite to that of the vector channel.

The first few terms of the model OPE in the vector channel read explicitly

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) - \frac{2m^6}{3Q^6} + \frac{2m^8}{Q^8} - \frac{22m^{10}}{5Q^{10}} + \frac{26m^{12}}{3Q^{12}} + \dots \quad (56)$$

The term of dimension eight is still available in OPE as expressed through the vacuum expectation value of local operators, even though it is very poorly known numerically [45], while higher order terms were never used in phenomenological applications. The expression in Eq. (50) shows also the actual scale of the expansion in the vector channel, $s_0 = 2m^2$. The scale Λ with $\Lambda \sim \Lambda_{\text{QCD}} \sim \Lambda_{\overline{\text{MS}}} \sim 350 \div 400 \text{ MeV}$ or $\Lambda^2 \sim 0.25m^2$ does not fit the scale of the power corrections in this model.

There is a numerical cancellation between the resonance and continuum contributions to the coefficients c_n of the first several terms while for higher order terms (large values of n) the scale $s_0 = 2m^2$ dominates. This cancellation is one of the reasons for the success of the Borel sum rules for the ρ meson in the vector channel. One can go very low in the Borel parameter M and still finds power corrections to be small if only power corrections up to dimension-six operators are included (with a nonvanishing gluon condensate there is also an additional cancellation between the dimension-four and dimension-six terms). Note that the expansion for Adler's function $D(Q^2)$ (or for the renormalized correlator function $\Pi(Q^2)$) converges for $Q^2 > 2m^2$. This is related to the simplicity of the analytic structure of the correlator in the model and therefore to the simplicity of the model spectrum. Therefore, if one is to believe that the shape of the spectrum is really close to that of the model in Eq. (48), the scale of power corrections should be close to the value $s_0 = 2m^2$. We discuss this issue later on in more detail for a variety of different models.

Next we consider the axial part of the correlator or the axial channel. Because of the presence of the pion, in this case the spectrum at low energies is drastically different from the one for the vector channel. All axial-vector resonances (with spin 1) have a finite mass. In the massless limit there is theoretically a Goldstone mode – corresponding to the observed pion – with spin zero contributing to the correlator of the axial-vector current (this is the reason why we call it more generally axial correlator or axial channel). The main mass scale is the mass m_{a_1} of the axial-vector meson a_1 which we express by $m_{a_1}^2 = 2m_A^2$ for further convenience. The underlying reason for this is Weinberg's relation: at some point one can identify the scale m_A with the scale in the vector channel which is given by the ρ meson mass m_ρ . A possible model for the spectrum in the axial channel reads

$$\rho_A(s) = m_A^2 \delta(s) + m_A^2 \delta(s - 2m_A^2) + \theta(s - 2m_A^2) \quad (57)$$

where the first term is the pion contribution, the second one is contribution of the a_1 meson, and the third represents the continuum. There is no gap left between the second resonance and the continuum. The correlator in the axial channel is given by

$$\Pi_A(Q^2) = \frac{m_A^2}{Q^2} + \frac{m_A^2}{2m_A^2 + Q^2} + \ln \left(\frac{\mu^2}{2m_A^2 + Q^2} \right), \quad (58)$$

the expansion at large Q^2 reads

$$\begin{aligned} \Pi_A(Q^2) &= \ln \left(\frac{\mu^2}{Q^2} \right) + \sum_{n=1}^{\infty} \left(-\frac{2m_A^2}{Q^2} \right)^n \left(\frac{1}{n} - \frac{1}{2}(1 + \delta_{n1}) \right) \\ &= \ln \left(\frac{\mu^2}{Q^2} \right) + \frac{4m_A^6}{3Q^6} - \frac{4m_A^8}{Q^8} + \frac{48m_A^{10}}{5Q^{10}} - \frac{64m_A^{12}}{3Q^{12}} + \dots \end{aligned} \quad (59)$$

where δ_{n1} is the Kronecker symbol. Here the contribution of the dimension-four operator is again zero while the dimension-six contribution is positive and larger than that in the vector channel, which is the case also in the (model independent) OPE (see Eq. (55)). While the continuum contribution (the logarithm and the $1/n$ part in the sum in Eq. (50)) remains the same, the factor $-1/2^{n-1}$ in case of the vector channel is replaced by $-1/2$ (for $n > 1$) in Eq. (59). Therefore, higher order power corrections for the vector channel are dominated by the continuum while for the axial channel they are dominated by the resonance contributions and are generally larger – the mass of the a_1 meson gives the scale both for the resonance contributions and the continuum threshold in this particular model. However, this cannot be

quantitatively checked at present because the numerical values of the higher order condensates are not known phenomenologically with sufficient accuracy.

It is instructive to compare the results for the model OPE expansions in the two channels,

$$\begin{aligned}\Pi_V(Q^2) &= \ln\left(\frac{\mu^2}{Q^2}\right) - \frac{2m_V^6}{3Q^6} + \frac{2m_V^8}{Q^8} - \frac{22m_V^{10}}{5Q^{10}} + \frac{26m_V^{12}}{3Q^{12}} + \dots \\ \Pi_A(Q^2) &= \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{4m_A^6}{3Q^6} - \frac{4m_A^8}{Q^8} + \frac{48m_A^{10}}{5Q^{10}} - \frac{64m_A^{12}}{3Q^{12}} + \dots\end{aligned}\quad (60)$$

The coefficients of the power corrections are different and the signs of corresponding terms are opposite. A numerical comparison is possible if one identifies the scales $m_A = m_V = m$ in both channels which is a rather reasonable phenomenological approximation because of Weinberg's relation $m_{a_1}^2 = 2m_\rho^2$. However, one should keep in mind that both models only capture the gross features of the spectra while the fine details (visible at high resolution in the energy) are different and of importance for the numerical magnitude of higher order power corrections. Indeed, higher order power corrections are sensitive to the fine details of the spectra and cannot be reliably determined within rough models. Therefore, while the models proposed in this subsection allow one to calculate power corrections up to any order, a reasonable accuracy is expected only for low orders in $1/Q^2$ – high-order power corrections can resolve the fine structure of the spectrum which is not caught by the gross models. It is expected that for the first few terms of the power expansion the accuracy is rather good while for high-order terms it can be only an order of magnitude approximation. Still such simple models based on the gross features of the spectrum are definitely useful for a general analysis. Higher order terms are more sensitive to details of the spectrum and can be predicted only with large errors. We discuss these issues below.

The moments corresponding to the model spectral density for the vector channel for $k = 0$ read (assuming $2m^2 < M_\tau^2$)

$$M_{0l} = 1 - \left(\frac{2m^2}{M_\tau^2}\right)^{l+1} \left(1 - \frac{l+1}{2^l}\right). \quad (61)$$

One observes that the perturbative contribution is represented by the first term on the right hand side of Eq. (61). The power corrections are given in what follows. The combined perturbative and power correction structure is a natural order for the direct s^l -moments. For

large l the contribution of the power corrections decreases and the moments are saturated by perturbation theory, i.e. if $m^2 \ll M_\tau^2$, the power corrections for the moments M_{0l} die out fast.

For mixed moments M_{kl} with $l = 0$ and arbitrary k we obtain (again for $2m^2 < M_\tau^2$)

$$M_{k0} = \left(1 - \frac{2m^2}{M_\tau^2}\right)^{k+1} + (k+1) \frac{2m^2}{M_\tau^2} \left(1 - \frac{m^2}{M_\tau^2}\right)^k. \quad (62)$$

The magnitude of these moments tends to zero for large values of k and definitely cannot be represented perturbatively. Indeed, according to the structure of OPE one should represent the moments as

$$M_{k0} = 1 + \Delta_{k0} \quad (63)$$

where Δ_{k0} gives the contribution of the power correction terms which are usually considered to be small. For large values of k , however, one has $M_{k0} \rightarrow 0$ (as Eq. (62) shows) and $\Delta_{k0} \rightarrow -1$ from Eq. (63), i.e. the moments are not given by an expansion near the perturbation theory result, $M_{k0} = 1$. The decomposition in Eq. (63) is therefore useless for large values of k because Δ_{k0} is not small. For the first few orders in k one has

$$\begin{aligned} \Delta_{00} &= \Delta_{10} = 0, & \Delta_{20} &= -2 \left(\frac{m^2}{M_\tau^2}\right)^3, & \Delta_{30} &= -8 \left(\frac{m^2}{M_\tau^2}\right)^3 + 8 \left(\frac{m^2}{M_\tau^2}\right)^4, \\ \Delta_{40} &= -20 \left(\frac{m^2}{M_\tau^2}\right)^3 + 40 \left(\frac{m^2}{M_\tau^2}\right)^4 - 22 \left(\frac{m^2}{M_\tau^2}\right)^5, \\ \Delta_{50} &= -40 \left(\frac{m^2}{M_\tau^2}\right)^3 + 120 \left(\frac{m^2}{M_\tau^2}\right)^4 - 132 \left(\frac{m^2}{M_\tau^2}\right)^5 + 52 \left(\frac{m^2}{M_\tau^2}\right)^6 \\ \Delta_{60} &= -70 \left(\frac{m^2}{M_\tau^2}\right)^3 + 280 \left(\frac{m^2}{M_\tau^2}\right)^4 - 462 \left(\frac{m^2}{M_\tau^2}\right)^5 + 364 \left(\frac{m^2}{M_\tau^2}\right)^6 - 114 \left(\frac{m^2}{M_\tau^2}\right)^7. \end{aligned} \quad (64)$$

If $m^2 \ll M_\tau^2$, the power corrections for the moments M_{k0} are still basically given by the lowest order term. If $2m^2$ is close to M_τ^2 as it is actually the case in τ decays (the conclusion is based on the model spectrum), power corrections for the moments M_{0l} are given by a single operator of dimension l and are relatively small while for the moments M_{k0} the power corrections are given by a linear combination of all operators up to the specified order. It is instructive to rewrite the k moments in terms of the actual scale $2m^2$. One has

$$\begin{aligned} \Delta_{00} &= \Delta_{10} = 0, & \Delta_{20} &= -\frac{1}{4} \left(\frac{2m^2}{M_\tau^2}\right)^3, & \Delta_{30} &= -\left(\frac{2m^2}{M_\tau^2}\right)^3 + \frac{1}{2} \left(\frac{2m^2}{M_\tau^2}\right)^4, \\ \Delta_{40} &= -\frac{5}{2} \left(\frac{2m^2}{M_\tau^2}\right)^3 + \frac{5}{2} \left(\frac{2m^2}{M_\tau^2}\right)^4 - \frac{11}{16} \left(\frac{2m^2}{M_\tau^2}\right)^5, \end{aligned}$$

$$\begin{aligned}
\Delta_{50} &= -5 \left(\frac{2m^2}{M_\tau^2} \right)^3 + \frac{15}{2} \left(\frac{2m^2}{M_\tau^2} \right)^4 - \frac{33}{8} \left(\frac{2m^2}{M_\tau^2} \right)^5 + \frac{13}{16} \left(\frac{2m^2}{M_\tau^2} \right)^6 \\
\Delta_{60} &= -\frac{35}{4} \left(\frac{2m^2}{M_\tau^2} \right)^3 + \frac{35}{2} \left(\frac{2m^2}{M_\tau^2} \right)^4 - \frac{231}{16} \left(\frac{2m^2}{M_\tau^2} \right)^5 + \frac{91}{16} \left(\frac{2m^2}{M_\tau^2} \right)^6 - \frac{57}{64} \left(\frac{2m^2}{M_\tau^2} \right)^7. \quad (65)
\end{aligned}$$

Looking at Eq. (65) one can conclude that keeping only the first contribution can give a completely wrong answer for the total result for Δ_{k0} at large k . Therefore, being expressed through the natural scale, i.e. the scale where the analytic properties of the correlator change, the two sets of moments behave differently. The direct moments have power corrections in accordance with the expectation that these corrections are powers of the relevant scale while the numerical magnitude of mixed moments is non-predictable on general grounds without a detailed knowledge of the spectrum at low energies. Note again that we neglect the effects of the anomalous dimensions of local operators in our simplified consideration based on a model spectrum which, however, is also a common practice in phenomenological applications.

The dominance of the resonances for the axial channel becomes obvious if we consider the moments. For $2m_A^2 < M_\tau^2$ the direct moments are given by

$$M_{00} = 1, \quad M_{0l} = 1 + \left(\frac{2m_A^2}{M_\tau^2} \right)^{l+1} \left(1 - \frac{l+1}{2} \right), \quad (66)$$

while the mixed moments for $l = 0$ read

$$M_{k0} = \left(1 - \frac{2m_A^2}{M_\tau^2} \right)^{k+1} + (k+1) \frac{m_A^2}{M_\tau^2} + (k+1) \left(1 - \frac{2m_A^2}{M_\tau^2} \right)^k \frac{m_A^2}{M_\tau^2}. \quad (67)$$

Because of the second term in M_{k0} which comes from the pion resonance at $s = 0$, the moments M_{k0} will not vanish for increasing values of k but will increase linearly in k . This is a feature worth emphasizing. For higher and higher values of k the moments M_{k0} in Eq. (62) go to zero if used for smooth spectral functions on the cut which is the case if the spectral densities are calculated in perturbation theory. However, the experimental shape of the spectrum is different, and the spectral density partially behaves like a distribution. The total pion contribution for example is concentrated at one point which is not affected by the decrease of the essential support of the measure with increasing k . Therefore, due to the normalization chosen the pion contribution grows with k . These two features of the experimental k moments, namely the decrease of the vector contribution and the increase of the axial contribution due to the pion

	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$	$l = 7$
$M_{0l}^{V \text{ ex}}$	0.98692	0.99021	0.99497	0.99777	0.99909	0.99964
$M_{0l}^{V(3)}$	0.98692	1.0000	1.0000	1.0000	1.0000	1.0000
$M_{0l}^{A \text{ ex}}$	1.0262	1.0196	1.0110	1.0055	1.0026	1.0011
$M_{0l}^{A(3)}$	1.0262	1.0000	1.0000	1.0000	1.0000	1.0000
M_{0l}^{ex}	1.0065	1.0049	1.0030	1.0016	1.0008	1.0004
$M_{0l}^{(3)}$	1.0065	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1: Moments M_{kl} for $k = 0$ for the vector channel and the axial channel model, using the exact result M_{0l}^{ex} as well as the power series expansion up to the third order term in $1/Q^2$, $M_{0l}^{(3)}$. We use the estimate $m_A^2 = m_V^2 = m_\rho^2 = 769.3 \text{ MeV}$ [3].

(as found in Eqs. (62) and (67)) are very essential for the comparison with the theoretical description of the τ -moments to be successful. Indeed, one has

$$M_{kl}^\tau = N_{kl}^\tau \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^{k+2} \left(\frac{s}{M_\tau^2}\right)^l \left(1 + \frac{2s}{M_\tau^2}\right) \frac{\rho(s)ds}{M_\tau^2} \quad (68)$$

(the normalization factor N_{kl}^τ is chosen so that $M_{kl}^\tau = 1$ for $\rho(s) = 1$). Remark that the normalized τ lepton decay rate R_τ as in Eq. (7) is just M_{00}^τ . The theoretical expression for the moments changes slowly with k , therefore the sum of vector and axial parts behave better than these two parts separately.

With the explicit models in Eqs. (48) and (57) at hand we can make a quantitative analysis. We assume that these models give the exact results at any order of the power expansion. Then we consider the moments and figure out when it is enough to keep only the three terms usually available in phenomenology in order to have a reasonable (given) accuracy. Table 1 shows the situation for the l moments. One sees that the perturbation theory contribution dominates the results. For both vector and axial correlators the accuracy of the three-term approximation is better than 2% and improves for large values of l as expected.

Table 2 shows the corresponding situation for the k moments. Here one sees the dominance of nonperturbative contributions to the vector and axial correlator results. For $k = 4$ the accuracy is already about 10% and deteriorates fast. For the sum of vector and axial correlators

	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$M_{k0}^{V \text{ ex}}$	0.98692	0.95745	0.91306	0.85716	0.79358	0.72591
$M_{k0}^{V(3)}$	0.98692	0.94766	0.86915	0.73830	0.54203	0.26725
$M_{k0}^{A \text{ ex}}$	1.0262	1.0851	1.1748	1.2902	1.4256	1.5761
$M_{k0}^{A(3)}$	1.0262	1.1047	1.2617	1.5234	1.9159	2.4655
M_{k0}^{ex}	1.0065	1.0213	1.0439	1.0737	1.1096	1.1510
$M_{k0}^{(3)}$	1.0065	1.0262	1.0654	1.1308	1.2290	1.3664

Table 2: Moments M_{kl} for $l = 0$ for the vector channel and the axial channel model, using the exact result M_{k0}^{ex} as well as the power series expansion up to the third order term in $1/Q^2$, $M_{k0}^{(3)}$.

We use the estimate $m_A^2 = m_V^2 = m_\rho^2 = 769.3 \text{ MeV}$ [3].

$l = 7$	-0.00	-0.01	-0.02	-0.02	-0.03	-0.03	-0.02	-0.00
$l = 6$	-0.01	-0.01	-0.02	-0.02	-0.02	-0.01	+0.00	+0.02
$l = 5$	-0.01	-0.02	-0.02	-0.02	-0.01	+0.00	+0.01	+0.02
$l = 4$	-0.01	-0.02	-0.02	-0.01	-0.00	+0.00	+0.01	+0.02
$l = 3$	-0.01	-0.01	-0.01	-0.01	-0.00	+0.01	+0.04	+0.10
$l = 2$	+0.02	+0.06	+0.14	+0.32	+0.66	+1.27	+2.33	+4.11
$l = 1$	+0.02	+0.02	-0.02	-0.21	-0.71	-1.88	-4.34	-9.24
$l = 0$	-0.01	-0.01	-0.02	-0.02	-0.01	+0.01	+0.00	-0.05
	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$

Table 3: Relative deviation of the moments $M_{kl}^{\tau(3)}$ for the power series up to the third order from the moments M_{kl}^τ for the full result containing the vector and the axial-vector channel model. We use the estimate $m_A^2 = m_B^2 = m_\rho^2 = 769.3 \text{ MeV}$ [3].

the accuracy of the three-term approximation is better as one can see from the last line of Table 2. This proves our statement about the behaviour of the sum of vector and axial part which can also be seen from the expansions in Eqs. (60). It is known up to the third order term from phenomenology where the corresponding values are given in Eqs.(54) and (55). Note, however, that the phenomenological results are obtained only in the factorization approximation for the vacuum expectation values of the four-quark operators. Thus, the pion contribution is crucial in keeping a reasonable accuracy for the sum of axial and vector correlators for the large k moments. This corresponds to the local duality picture.

Our general arguments are becoming more transparent if we consider mixed moments M_{kl} with non-zero values of l . This is done in Table 3 where we give ratios

$$\frac{M_{kl}^\tau - M_{kl}^{\tau(3)}}{M_{kl}^\tau} \quad (69)$$

for the τ -moments defined in Eq. (68). Then only τ -moments for large values of l and small values of k show perturbative behaviour. The bottom line in Table 3 is drastically different from the rest of the table, the reason being that the massless pion only contributes to moments with $l = 0$. This makes the large k moments still reasonably precise using only third-order power corrections.

4.3 Modifications of the model for the axial channel

Here we discuss the choice of the spectrum in the axial channel. One can suggest forms differing from those given in Eq. (57). The reason for this freedom is that we have two resonances which are essential at low energies – the pion and a_1 meson – that bring in four free parameters: the two residues of the resonances, the mass of the a_1 meson, and the beginning of the continuum. Therefore, two requirements for the vanishing of the power corrections $1/Q^2$ and $1/Q^4$ are not sufficient to uniquely fix the low-energy spectrum in a simple way (with only one scale). There can be many different additional requirements all reasonably close to the experimental data. Note that the a_1 meson shows up as a broad resonance which provides additional freedom in parameterizing its contribution to the spectrum. For a precision application the best choice is to fix the pion residue and the mass of the a_1 meson to their experimental values while leaving

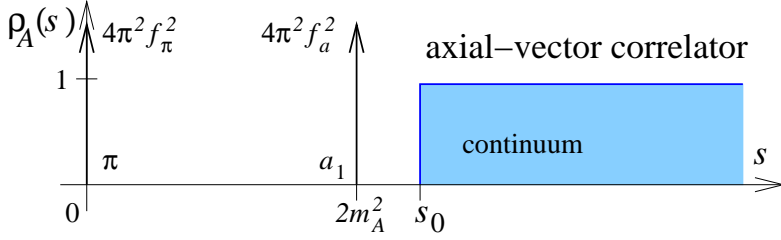


Figure 5: Spectral density for the modified axial-vector channel model given by Eq. (72).

its residue and the continuum threshold to be determined from OPE. In fact, the experimental value of f_π and simple duality requirements give a rather good description of the axial channel.

Another possibility is to fix the residue of the a_1 meson to be $2m_A^2$ as in the vector channel (see Eq. (48)). Then the spectrum in the axial-vector channel reads

$$\rho_A(s) = 2(\sqrt{2} - 1)m_A^2\delta(s) + 2m_A^2\delta(s - 2m_A^2) + \theta(s - 2\sqrt{2}m_A^2) \quad (70)$$

which also leads to a result in which the coefficients of the first two power corrections vanish. The third coefficient of the large Q^2 expansion of the axial correlator now reads

$$c_3^A = \frac{8}{3}(3 - 2\sqrt{2}) = 0.457528 < \frac{2}{3} \quad (71)$$

(cf. Eq. (60)) which is smaller than in the model given by Eq. (57). It is also smaller than in the vector channel which can lead to a (slight) contradiction with the phenomenology if the scales m_V and m_A are identified and factorization of the four-quark condensate is taken as being exact. Still it is numerically acceptable from the phenomenological point of view.

Having fixed the mass of the a_1 meson to $2m_A^2$ one has a one-parameter family of spectra in the axial-vector channel

$$\rho_A(s) = 4\pi^2 f_\pi^2 \delta(s) + 4\pi^2 f_a^2 \delta(s - 2m_A^2) + \theta(s - s_0) \quad (72)$$

with s_0 being a free parameter. Then

$$4\pi^2 f_a^2 = \frac{s_0^2}{4m_A^2}, \quad 4\pi^2 f_\pi^2 = s_0 - \frac{s_0^2}{4m_A^2}. \quad (73)$$

The spectral density of is modified model for the axial channel is shown in Fig. 5. One constraint for the construction of the model is $s_0 > 2m_A^2$. However, this constraint is quite weak. One can actually admit a resonance as a bump in the continuum which is almost the case for

the a_1 meson. The positivity constraint for f_π^2 is given by $s_0 < 4m_A^2$ and has to be taken seriously. Note that at $s_0 = 4m_A^2$ the pion decouples and does not contribute ($f_\pi = 0$) to the spectrum of the correlator which is not the case in reality. For some values of s_0 in the interval $2m_A^2 < s_0 < 4m_A^2$ the spectrum can be fixed by taking the experimental value for f_π . A possible additional constraint is given by the value of the dimension-six condensate. This was used for the finite-energy sum rule analysis and turned out to be rather successful phenomenologically [41].

The residue of the a_1 meson and the continuum threshold for the model in Eq. (72) (as being expressed in terms of the pion decay constant f_π and the mass m_{a_1} of the a_1 meson) are given by

$$s_0 = 2m_A^2 \left(1 + \sqrt{1 - 4\pi^2 f_\pi^2 / m_A^2} \right), \quad 4\pi^2 f_a^2 = s_0 - 4\pi^2 f_\pi^2. \quad (74)$$

The consistency constraint reads

$$4\pi^2 f_\pi^2 / m_A^2 \leq 1. \quad (75)$$

Experimentally we have $4\pi^2 f_\pi^2 \approx m_A^2$ which is close to the model without a gap. Our conclusions about moments are valid for these models as well.

The considerations obtained for the simple models lead to the conclusion that the structure of power corrections for the moments with weight $(1 - s/M_\tau^2)^k$ is strongly infrared sensitive and nonperturbative. One can expect that this conclusion will be retained for a realistic model (QCD), i.e. that power corrections are very important and that the perturbation theory contribution is strongly suppressed for the moments with weight $(1 - s/M_\tau^2)^k$. This was to be expected because these moments are saturated by resonances.

4.4 A note on theoretical models for power corrections

For the theoretical analysis of large k moments one needs all power corrections. These are available only in specific models. In the previous parts of this section we have discussed models for the spectra based on the experimentally measured shape of the spectra while taking into account only main characteristic features of them. However, there are also theoretical results for the spectra that are usually obtained in simplified models of quantum field theory and which, as one believes, can have some relation to the low-energy hadron spectra in QCD.

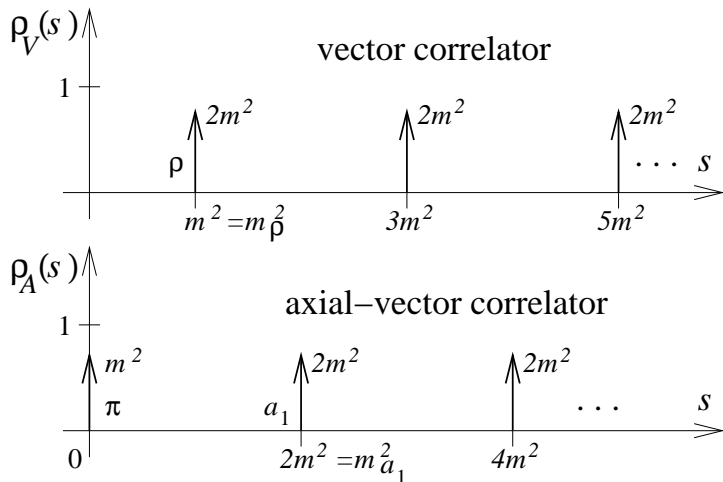


Figure 6: Spectral density of the narrow resonance models for the vector and axial-vector channel as given in Eqs. (78) and (82).

The old example is two-dimensional QED (QED₂) which was solved by Schwinger [46]. The OPE for current correlators in QED₂ was analyzed in Ref. [47]. In this model the scale of power corrections is given by the dimensional charge e (the model is super-renormalizable). Both vector and axial-vector currents have the same scale e^2/π , in fact they are given by a free massive field and are trivial in this respect. The photon acquires a mass due to the anomaly. The interesting channels are that of the scalar and pseudoscalar currents, the correlators for which are known exactly and give an example of a really nontrivial spectrum [47].

There are some results concerning all-order perturbation theory diagrams calculations in a ladder approximation [48]. These results can be used for constructing realistic four-dimensional models where all power corrections are explicitly known. The sum rule analysis of OPE in such models can reveal some features different from the standard phenomenology [49].

A popular model is the $1/N_c$ approximation both in four and two-dimensional Abelian and non-Abelian theories (see e.g. Ref. [50, 51]) where the spectrum is given by infinitely narrow resonances, as shown in Fig. 6. This type of spectrum is predicted by a Regge analysis or by local duality arguments. A recent analysis along this line was presented in Refs. [52, 53].

The model of narrow resonances inspired by the 't Hooft model was analyzed in QCD on the basis of local duality [43]. The spectrum was studied within the local duality approach

where one has [43]

$$m_n^2 = (2n+1)m_\rho^2, \quad f_n^2 = 2m_\rho^2 \quad n = 0, 1, \dots \quad (76)$$

for the vector channel and where m_ρ is the mass of the the ground state ρ meson. For the axial-vector channel including a massless pion as Goldstone boson (axial channel) the result of the local duality approach coupled with the $1/N_c$ approximation reads [43]

$$m_n^2 = nm_{a_1}^2, \quad f_0^2 = 4\pi^2 f_\pi^2 = \frac{1}{2}m_{a_1}^2 = m_\rho^2, \quad f_n^2 = 2m_\rho^2 = m_{a_1}^2, \quad n = 1, 2, \dots \quad (77)$$

where m_{a_1} is the mass of the ground state a_1 meson. The structure of the spectrum reflects the classical results on chiral symmetry and Weinberg's relations for axial-vector and vector meson masses which is realized if one identifies the scales in both channels, $m_{a_1}^2 = 2m_\rho^2 = 2m^2$. This identification leads to a simplified picture where two chains of resonances are simply shifted by an amount $m_\rho^2 = m^2$. This is the gross structure of the spectrum. Experimental results differ quite a bit from this picture, but such details can be accounted for by using the OPE [41].

Within the model based on the local duality approach the summation of all resonances results in the vector correlator

$$\begin{aligned} \Pi_V(Q^2) &= \sum_{n=0}^{\infty} \frac{f_n^2}{m_n^2 + Q^2} = \sum_{n=0}^{\infty} \frac{2m^2}{(2n+1)m^2 + Q^2} = \\ &= \sum_{n=0}^{\infty} \frac{1}{n + (Q^2 + m^2)/(2m^2)} = -\psi\left(\frac{Q^2 + m^2}{2m^2}\right) + \text{subtractions.} \end{aligned} \quad (78)$$

Here $\psi(z)$ is the digamma function. To find the link to the OPE we consider the large Q^2 behaviour of the model correlator in Eq. (78). The large z expansion of the digamma function $\psi(z)$ reads

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{k=1}^{n-1} \frac{B_{2k}}{2kz^{2k}} + O\left(\frac{1}{z^{2n}}\right) \quad (79)$$

where B_{2k} are the Bernoulli numbers

$$B_{2k} = \frac{(-1)^{k-1}(2k)!}{2^{2k-1}\pi^{2k}}\zeta(2k), \quad B_{2k+1} = 0 \quad (80)$$

and $\zeta(2k)$ is the Riemann ζ function. This expansion is an asymptotic expansion. For the correlator one then obtains

$$\Pi_V(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) - \frac{m^4}{6Q^4} + 0\frac{m^6}{Q^6} + \frac{7m^8}{60Q^8} + 0\frac{m^{10}}{Q^{10}} + \dots \quad (81)$$

where the renormalization scale μ comes in through the subtraction term. Note that the expression in Eq. (81) is quite different from the result obtained in Eq. (50).

In the axial channel the model based on local duality (with parameters from Eq. (77)) leads to the expression

$$\begin{aligned}\Pi_A(Q^2) &= \sum_{n=0}^{\infty} \frac{f_n^2}{m_n^2 + Q^2} = \frac{f_0^2}{Q^2} + \sum_{n=1}^{\infty} \frac{m_{a_1}^2}{m_{a_1}^2 n + Q^2} = \\ &= \frac{m_{a_1}^2}{2Q^2} + \sum_{n=1}^{\infty} \frac{1}{n + Q^2/m_{a_1}^2} = -\frac{m_{a_1}^2}{2Q^2} - \psi\left(\frac{Q^2}{m_{a_1}^2}\right) + \text{subtractions}\end{aligned}\quad (82)$$

and the large Q^2 behaviour reads

$$\Pi_A(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{m_{a_1}^4}{12Q^4} + 0\frac{m_{a_1}^6}{Q^6} - \frac{m_{a_1}^8}{120Q^8} + 0\frac{m_{a_1}^{10}}{Q^{10}} + \dots \quad (83)$$

In this case all power corrections of dimension $2(2k+1)$ vanish because of $B_{2k+1} = 0$. This contradicts the results of OPE since there is no explicit (symmetry) reason for such a property. At moderate orders of n the structure of the expansion of Eq. (82) as given in Eq. (83) is inconsistent with the asymptotic expansion expected from the OPE (see Eqs. (48), (49), (50), (58), and (59)) and, therefore, is not supported by phenomenology. Though the model of infinitely narrow resonances is rather attractive, this inconsistency was the reason that this model was not analyzed quantitatively in the literature and was instead considered only as a rough approximation for the first few moments (duality relation). Indeed, the zeroth moment is correct (no $1/Q^2$ terms) and the first moment is correct in the approximation of a vanishing gluon condensate contribution but the vanishing of the dimension-six terms are definitely unacceptable phenomenologically. The high order $1/Q^2$ behaviour can be easily corrected by shifting the parameters of the first one or two resonances. But then the simplicity of the functional form of this model with its one-parameter dependence for the whole spectrum is lost.

The qualitative difference of the models given in Eqs. (81) and (83) from the previous case with a continuum contribution, however, is the analytic structure of the correlators. In the narrow resonance model one only has a single dimensional parameter m^2 and one would expect the power corrections to behave as $(m^2/Q^2)^n$ with m^2 determining the scale. However, the coefficients of the power corrections given by the Bernoulli numbers grow quite fast as the large k behaviour demonstrates,

$$B_{2k} = 4(-1)^{k-1} \sqrt{\pi k} \left(\frac{k}{\pi e}\right)^{2k} (1 + o(1)) \quad (84)$$

(the notation $o(x^p)$ indicates contributions which fall off to zero faster than x^p for $x \rightarrow 0$). Thus, the expansion in Eqs. (81) and (83) is only asymptotic. The reason for the divergent nature of the series of power corrections is that for the models given in Eqs. (78) and (82) there are poles located arbitrarily far away from the origin $Q^2 = 0$. In this model, therefore, there is in fact an infinite number of scales nm^2 for positive integers n . This is the reason why perturbation theory does not work even at sufficiently large s . But if one approximates a chain of resonances by a continuum starting from some threshold s_0 , one instead gets a finite radius of convergence of the order s_0 , $Q^2 > s_0$ and a well working perturbation theory picture at $s > s_0$.

Let us compare the behaviour of moments for such models. Because of the complicated argument of the ψ function in Eq. (78) for the vector correlator, power corrections in the form of an expansion in $1/Q^2$ are not directly given by the expansion of the ψ function. This makes the calculation of an arbitrary term for the vector correlator clumsy. In case of the axial correlator this expansion is straightforward as one can see from Eq. (82). Therefore, we shall only analyze the axial channel but will give a functional relation that holds for both correlators in these models. Identifying $2m^2$ with $m_{a_1}^2$, $m_{a_1}^2 = 2m^2$ one finds

$$\Pi_A(Q^2) = \Pi_V(Q^2 - m^2) - \frac{m^2}{Q^2} \quad (85)$$

or, when rewritten in the Minkowskian domain,

$$\Pi_A(q^2) = \Pi_V(q^2 + m^2) + \frac{m^2}{q^2}. \quad (86)$$

This relation is inspired by particular models for the spectra but can be checked for experimental data as well (for Adler's functions which are independent of subtractions).

The narrow resonance model for the vector and axial channel does not display the difference between low and high energies. The only criterion of perturbation theory calculability is the length of the averaging interval while its position in terms of energies is almost unimportant. This is a natural feature of the translation invariance of the spectra in these models. We now discuss the moments in the axial channel in more detail. In order to obtain simple estimates we take the averaging interval to be $5m^2$ (in “real life” the averaging interval is given by $M_\tau^2 = 5.15m_\rho^2$ but $5m^2 = 5m_\rho^2$ is a good approximation). We check the quality of the decomposition

vector	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 20$
Δ_{0n}^V	+0.00	+0.00	-0.03	-0.06	-0.09	-0.13	-0.40	-0.82
Δ_{n0}^V	+0.00	+0.00	-0.03	-0.06	-0.09	-0.13	-0.40	-0.82

axial	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n = 20$
Δ_{0n}^A	+0.00	-0.04	-0.04	-0.08	-0.13	-0.19	-0.53	-0.90
Δ_{n0}^A	+0.00	+0.04	+0.08	+0.16	+0.26	+0.39	+1.23	+3.20

Table 4: Deviations Δ_{kl}^V and Δ_{kl}^A (see Eq. (87)) of the moments for the narrow resonance vector and axial channel model with respect to the parton contribution $\Delta_{kl}^{V/A} = 1$ for $k = 0$ or $l = 0$.

from the perturbation theory point of view, i.e. we write the result in the form

$$M_{kl} = 1 + \Delta_{kl}. \quad (87)$$

The direct l moments obtained for the power expansion in Eq. (83) are (for $l > 0$) given by

$$M_{0l} = 1 + (l + 1) \left(\frac{m_a^2}{M_\tau^2} \right)^{2l} B_{2l}. \quad (88)$$

For large values of l the decomposition as given in Eq. (87) is useless because the coefficients are growing very fast and thus the second term becomes large (see Eq. (84)). Still, for odd moments the decomposition is precise because then only one power correction is picked up and its contribution vanishes. However, the result based on the asymptotic value $M_{0l} = 1$ is quite different from the exact answer

$$M_{0l} = \left(\frac{m_a^2}{M_\tau^2} \right)^{2l} + \left(\frac{2m_a^2}{M_\tau^2} \right)^{2l}. \quad (89)$$

The reason for this is that the asymptotic behaviour is too crude to resolve the position of the poles near the origin. This can be seen from the expression for the vector channel where the appropriate expansion parameter for power corrections is $1/(Q^2 + m^2)$ instead of $1/Q^2$. In other words, the scale M_τ^2 is too small to be able to reflect any information about remote poles which give contributions to the asymptotic behaviour at large Q^2 .

For the mixed moments in the axial channel our results are given in the lower part of Table 4. We give only numerical results because it is not straightforward to represent the

decomposition analytically. The message is clear: both systems of moments – direct and mixed – are nonperturbative which is expected for this model where for any energy region the spectrum has nonperturbative character (close to a singularity).

An even more symmetric picture emerges in the vector channel. In this case one does not see such a drastic difference between moments as should be expected from the perturbation theory point of view: the situation is symmetric, the scale of “bumpiness” is the same at the both ends of the spectra. This fact makes both systems of moments almost equivalent. If we take the average interval $6m^2$, the result for exact moments is the same for both systems of moments because of the relation

$$(1 - 1/6)^n + (1 - 3/6)^n + (1 - 5/6)^n = (1/6)^n + (3/6)^n + (5/6)^n, \quad (90)$$

as shown in the upper part of Table 4. In general, for the vector current there is just an exact symmetry of the model spectrum $s \rightarrow (2km^2 - s)$ which makes the result of the calculation for both systems of moments simply equal. Still the contribution of power corrections is different for different systems of moments and, moreover, does not allow to restore the shape of the exact spectrum.

This symmetry of the simplified model spectrum is definitely violated in the realistic phenomenological spectrum, making the region of high energy essentially different from the low energy domain. This is because of the asymptotic freedom in QCD where at large energies the spectrum becomes a more and more continuous function because the features of the resonances as distributions becomes less and less pronounced. This also means that the model for the spectrum as an infinite chain of infinitely narrow resonances does not properly incorporate the asymptotic freedom of QCD in a sense of scale invariance violation.

We should note here that if there is a new situation at large energies (as the production of heavy quarks near their threshold) one should do the analysis as if the energy were counted from the respective thresholds. It is not the absolute value of the energy that matters but the distance from the strong interaction region.

As a last remark we stress that even the formal knowledge of the asymptotic series in $1/Q^2$ does not allow one to restore the spectrum point-wise. This reflects the same situation as in phenomenological analyses, namely that the determination of the spectral density is

an improperly posed problem (inverse problem). In our particular example one can actually observe this feature. For the axial channel one can still figure out the spectrum from the coefficients by restoring the full function but for the vector channel it is not straightforward. There is actually no systematic method for doing so. Taking just the “leading” asymptotics for the coefficients one can easily obtain identical results for the axial and vector channels which is definitely an unwelcome result as regards low-energy phenomenology. Therefore, the technique which could produce such a result (like the Borel summation of the “leading” asymptotics) can be quite misleading for calculating the low-energy spectrum as we know it from phenomenology.

5 On the determination of the strange quark mass

In case of the $(\bar{u}s)$ current the mass of the strange quark should be taken into account. This was discussed recently in Refs. [6, 7, 8]. The conclusions we have previously drawn about moments with weight function $(1 - s/M_\tau^2)^k$ for the massless case remain the same. We comment only on a new feature of the massive correlators with the s quark mass m_s considered as a small correction. The occurrence of the strange quark mass leads to an additional freedom in the choice of the scalar function for the momentum-type analysis because the correlator is no longer transverse (unlike in Eq. (3)). In the literature there are different suggestions about which of the correlator parts $\Pi_q(q^2)$ and $\Pi_g(q^2)$ in Eq. (2) is better suited for the extraction of the strange quark mass. It is worth to have a closer look at the situation. While in present calculations radiative corrections are taken into account, the important term is still the leading order contribution. Because m_s is small, this gives a correction which should be taken at the lowest order available in order to have a reasonable accuracy. The leading order spectral density for the current $\bar{u}\gamma_\mu(\gamma_5)s$ related to the part $\Pi_q(q^2)$ of the correlator in Eq. (2) reads

$$\rho_q(s) = \left(1 - \frac{3m_s^4}{s^2} + \frac{2m_s^6}{s^3}\right) \theta(s - m_s^2). \quad (91)$$

This spectral density is perturbatively insensitive to m_s^2 because at large s the expression in Eq. (91) starts with a term proportional to m_s^4 . A term of order m_s^2 appears only after the integration of the spectrum over s with weight functions which are hard (nonvanishing) at the origin (at $s \sim m_s^2$ or $s = 0$ for small m_s^2). Moments given by powers of s (as s^l , $l > 0$) have no

contribution of order m_s^2 . Indeed, for the zeroth order direct moment one finds

$$M_{00}^q = \int_{m^2}^{M_\tau^2} \frac{\rho_q(s)ds}{M_\tau^2} = 1 - 3\frac{m_s^2}{M_\tau^2} + 3\left(\frac{m_s^2}{M_\tau^2}\right)^2 - \left(\frac{m_s^2}{M_\tau^2}\right)^3 \quad (92)$$

where a term of order m_s^2 occurs. On the other hand, for the first moment

$$M_{01}^q = 2 \int_{m^2}^{M_\tau^2} \left(\frac{s}{M_\tau^2}\right) \frac{\rho_q(s)ds}{M_\tau^2} = 1 + 3\left(\frac{m_s^2}{M_\tau^2}\right)^2 - 4\left(\frac{m_s^2}{M_\tau^2}\right)^3 + 6\left(\frac{m_s^2}{M_\tau^2}\right)^2 \ln\left(\frac{m_s^2}{M_\tau^2}\right) \quad (93)$$

there is only a logarithmically enhanced m_s^4 contribution. For higher moments there are only higher order corrections, e.g. for $l = 2$ one has

$$M_{02}^q = 3 \int_{m^2}^{M_\tau^2} \left(\frac{s}{M_\tau^2}\right)^2 \frac{\rho_q(s)ds}{M_\tau^2} = 1 - 9\left(\frac{m_s^2}{M_\tau^2}\right)^2 + 8\left(\frac{m_s^2}{M_\tau^2}\right)^3 - 6\left(\frac{m_s^2}{M_\tau^2}\right)^3 \ln\left(\frac{m_s^2}{M_\tau^2}\right). \quad (94)$$

Therefore, perturbative s^l -moments of $\Pi_q(q^2)$ have no sensitivity to m_s^2 . This is also expected for this correlator part because the mass corrections start from power corrections and lead to no cut which is a perturbative singularity in QCD and reliable for large s . Note in passing that this is also the reason why the perturbation theory expansion for the coefficient function of m_s^2 in the correlator part $\Pi_q(q^2)$ is known by one term less than that for the part $\Pi_g(q^2)$: for the correlator part $\Pi_g(q^2)$ only divergent parts of the four-loop diagrams are necessary while for the part $\Pi_q(q^2)$ one has to calculate the finite contributions of the four-loop diagrams which is beyond the scope of present computational algorithms.

The afore mentioned structure of the spectral density also means that there is no way to improve the theoretical accuracy of the contribution of the mass by making the corresponding coefficient function more perturbative: as soon as one tries to make the coefficient function multiplying m_s^2 more perturbative by introducing an additional suppression by a power of s (or by increasing l for the modified moments), the contribution of the mass becomes parametrically smaller, so instead of m_s^2 one has m_s^4 in the leading order. When the anomalous dimension of the mass is taken into account, the power s then leads to an additional suppression by a power of α_s .

On the other hand, the $(M_\tau^2 - s)^k$ -moments all have the same contribution of order m_s^2 given simply by the zero order direct moment M_{00}^q of Eq. (92),

$$M_{k0}^q = (k+1) \int_{m^2}^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^k \frac{\rho_q(s)ds}{M_\tau^2} = 1 - 3(k+1)\frac{m_s^2}{M_\tau^2} + o\left(\frac{m_s^2}{M_\tau^2}\right). \quad (95)$$

This amplitude is used in Ref. [8] where a better stability in the $\overline{\text{MS}}$ scheme was detected.

In contrast to $\rho_q(s)$, for the spectrum $\rho_g(s)$ of the amplitude $\Pi_g(q^2)$ in Eq. (2) one has

$$\rho_g(s) = \left(-s + \frac{3m_s^2}{2} - \frac{m_s^6}{2s^2} \right) \theta(s - m_s^2) \quad (96)$$

which is sensitive to m_s^2 for large s . Any perturbative moment of this amplitude has m_s^2 contributions and can be used in the perturbation theory analysis of m_s^2 corrections. Namely,

$$M_{0l}^g = (l+1) \int_{m_s^2}^{M_\tau^2} \left(\frac{s}{M_\tau^2} \right)^k \rho_g(s) \frac{ds}{M_\tau^2} = -M_\tau^2 \left\{ \frac{l+1}{l+2} - \frac{3m_s^2}{2M_\tau^2} + o\left(\frac{m_s^2}{M_\tau^2}\right) \right\}. \quad (97)$$

Therefore, the choice of the correlator part $\Pi_g(q^2)$ is definitely more reliable perturbatively for analyzing mass corrections, a fact that is also discussed in Refs. [6, 7]. However, the experimental situation is again less favourable to the choice of $\Pi_g(q^2)$ in comparison to $\Pi_q(q^2)$. The correlator part $\Pi_q(q^2)$ contains a contribution of the K meson which is well pronounced and can be easily seen while the amplitude $\Pi_g(q^2)$ contains only spin-one states and is poorly measured. In order to benefit from a better experimental accuracy, the use of the amplitude $\Pi_q(q^2)$ is favoured, but this choice leads to sum rules which are rather similar to the pseudoscalar sum rules (see e.g. Ref. [54]) and does not provide new aspects for the strange quark mass determination specific to τ decays.

6 Integration of running effects to all orders

In this section we discuss the technique of resumming the effects of renormalization group caused running of the strong coupling and other parameters (simply called “running” in the following) by taking the integration path along the contour of a circle in the complex q^2 -plane. As was discussed earlier in Sec. 3, the integration along the contour of a circle in the complex q^2 -plane is completely equivalent to the integration along the cut if finite-order perturbation theory expressions for the polarization functions are used. This is the direct mathematical consequence of the analytic properties of the functions $\ln^p(-M_\tau^2/q^2)$ for positive integer values p . However, if the renormalization group improved polarization function is used for the integration, some new features appear. This was also discussed in Sec. 4. The reason for this is that the analytic properties of resummed polarization functions are different from those in finite-order perturbation theory.

The integration along the contour including a full renormalization group resummation is now the most popular technique of accounting for the running of perturbative quantities: it efficiently resums an infinite number of terms generated by the evolution of the coupling constant [55, 56]. Clearly it also includes some nonperturbative features when an infinite number of terms is resummed which means that this also constitutes a special recipe for resummation of perturbation series [57]. The formulae for the integration along the circle are known in a closed form as one-fold integrals but the final integration itself is usually done numerically. An analysis of the analytical form of these integral expressions is useful for clarifying the real content of such a resummation technique. In doing this we discuss how our general considerations given in the previous sections work within this resummation technique. It should definitely give the results confirming the physical considerations (effective scale structure) qualitatively, otherwise the technique of resummation would be irrelevant for physical applications. As we shall see, the resummation technique fully agrees with the conclusions about perturbative calculability and asymptotic structure of the moments found in the analysis on the cut in the previous sections.

6.1 Quantities on the circle

As in the case of the analysis on the physical cut, our main concern is to account for the running. Therefore, the introduction of an effective coupling is helpful. Here we introduce an effective coupling in the Euclidean domain which requires some new notation.

We start with Adler's function in Eq. (12), introducing the effective strong coupling $\alpha_E(Q^2)$,

$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = 1 + \frac{\alpha_E(Q^2)}{\pi}. \quad (98)$$

The effective coupling $\alpha_E(Q^2)$ obeys the renormalization group equation (see e.g. [58])

$$Q^2 \frac{d}{dQ^2} \left(\frac{\alpha_E(Q^2)}{\pi} \right) = \beta(\alpha_E(Q^2)). \quad (99)$$

For the considerations we are aiming at it is enough to use only leading order in the β -function which contains already the bulk of the whole effect. Effects due to higher order corrections of the β -function are really small and do not change the basic picture. They only slightly affect the conclusions numerically [59]. Therefore, we consider the renormalization group equation

$$Q^2 \frac{d}{dQ^2} \left(\frac{\alpha_E(Q^2)}{\pi} \right) = -\beta_0 \left(\frac{\alpha_E(Q^2)}{\pi} \right)^2. \quad (100)$$

The resummed correlation function reads

$$\Pi(Q^2) = \ln\left(\frac{\mu^2}{Q^2}\right) + \frac{1}{\beta_0} \ln\left(\frac{\alpha_E(Q^2)}{\pi}\right) + \text{subtractions} \quad (101)$$

where

$$\alpha_E(Q^2) = \frac{\alpha_\tau}{1 + (\beta_0 \alpha_\tau / \pi) \ln(Q^2 / M_\tau^2)} \quad (102)$$

with $\alpha_\tau = \alpha_E(M_\tau^2)$. Taking this into account and parameterizing the contour by $Q^2 = M_\tau^2 e^{i\varphi}$ one obtains

$$\Pi(M_\tau^2 e^{i\varphi}) = -i\varphi - \frac{1}{\beta_0} \ln(1 + i\beta_0 \alpha_\tau \varphi / \pi) + \text{subtractions} \quad (103)$$

where appropriate subtractions are added. The first term is the parton contribution which is independent of α_τ . This term is readily be taken into account. We therefore concentrate on the contribution coming from the second term which is α_τ dependent.

The analysis of this second part can now be done for the moments M_{kl} , given by

$$\begin{aligned} M_{kl} &= 1 + m_{kl} = \frac{(-1)^l (k+l+1)!}{2\pi i k!l!} \oint_{|z|=1} \Pi(M_\tau^2 z) (1+z)^k z^l dz = \\ &= \frac{(-1)^l (k+l+1)!}{2\pi k!l!} \int_{-\pi}^{\pi} \Pi(M_\tau^2 e^{i\varphi}) (1 + e^{i\varphi})^k e^{i(l+1)\varphi} d\varphi. \end{aligned} \quad (104)$$

Taking the polarization function as given in Eq. (103), the first part leads to the parton result $M_{kl} = 1$ while the second, α_τ -dependent part gives the moment m_{kl} where

$$m_{kl} = \frac{(-1)^{l+1} (k+l+1)!}{2\pi \beta_0 k!l!} \int_{-\pi}^{\pi} (1 + e^{i\varphi})^k e^{il\varphi} \ln(1 + i\beta_0 \alpha_\tau \varphi / \pi) e^{i\varphi} d\varphi. \quad (105)$$

Note that the estimate of the saturation region for the moments given by integrals like the one in Eq. (105) is a bit trickier than in the case of the integration on the cut. On the cut the measure is positive and is normalized to 1 for any type of moments with $n \geq 0$,

$$(n+1) \int_0^1 x^n dx = 1 \quad \text{and} \quad (n+1) \int_0^1 (1-x)^n dx = 1 \quad (106)$$

which makes the estimate of integrals

$$(n+1) \int_0^1 x^n f(x) dx \quad (107)$$

for continuous (and rather smooth) functions $f(x)$ straightforward. In the present case, however, the measure of the integration region is oscillating and vanishes for constant functions ($k \geq 0, l \geq 0$),

$$\int_{-\pi}^{\pi} (1 + e^{i\varphi})^k e^{il\varphi} e^{i\varphi} d\varphi = 0. \quad (108)$$

Therefore, some additional care in the consideration of essential regions of integration is necessary for the formulation of moments on the contour. Note that we have already encountered such a problem while analyzing the moments of the narrow resonance contributions. Therefore, the intuition gained from the situation with continuous functions generally does not work for distributions.

6.2 Techniques for the calculation of contour moments

Because explicit functions are given in Eqs. (101), (103), (104), and (105), the discussion of the representation and properties of the moments on the contour is a pure mathematical problem. Since that moments on the circle (as in Eq. (105)) are just Fourier coefficients of the correlation functions $\Pi(Q^2)$, the technique for working with these is well-known. One finds similarities to the evaluation of sunset diagrams for the case when the problem is reduced to a Fourier transform.

For $\beta_0\alpha_\tau < 1$ the moments in Eq. (105) can be expanded in a convergent series in α_τ . The finite radius of convergence within this technique of resummation is a general feature which persists in higher orders of the β -function and was studied in some detail [59]. The convergence radius decreases when higher orders of the β -function are included. If we compare the value of the effective coupling in τ decays (not to be mistaken with the coupling within the $\overline{\text{MS}}$ -scheme)

$$\frac{\alpha_\tau^{\text{exp}}}{\pi} = 0.14 \tag{109}$$

with

$$\frac{1}{\pi\beta_0} = \frac{4}{9\pi} = 0.1415\dots, \tag{110}$$

the relation $\beta_0\alpha_\tau < 1$ is still valid, though almost marginally. However, the exact expression given in Eq. (105) without expansion of the logarithm provides an analytic continuation beyond the convergence radius even when α_τ lies outside the convergence radius.

We first consider the basic moment m_{00} . The discussion of this basic moment leads to the technically simplest (shortest) expressions while containing already all features of the general case. The generalization to other moments (and types of moments) is then straightforward.

The most direct way of calculation is to expand the logarithm in the integrand in Eq. (105) in a series in α_τ which gives nothing new in comparison with the finite-order perturbation

theory case. However, one can generate all terms of the series because the function is known. The series obtained by this procedure represents an analytic function of the coupling constant at the origin. Therefore, for small α_τ this series can be used for a numerical evaluation of the moments. This procedure may not be the best one concerning economy of evaluation since many terms may be needed to determine values of α_τ close to the convergence boundary. A finite piece of the convergent series gives different accuracy and can also completely misrepresent the function if α_τ is beyond the convergence radius (which is not so dramatic for positive values of α_τ , though).

One can proceed with the analysis of the moments in a different way by constructing just an efficient computational scheme. Integrating n times by parts one obtains (we remind the reader that we use the zeroth order moment for simplicity)

$$m_{00} = \frac{1}{\pi\beta_0} \left\{ \phi + \sum_{j=1}^{n-1} (j-1)! \left(\frac{\beta_0\alpha_\tau}{\pi r} \right)^j \sin(j\phi) + \frac{(n-1)!}{2} \left(\frac{\beta_0\alpha_\tau}{\pi} \right)^n \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0\alpha_\tau\varphi/\pi)^n} \right\}. \quad (111)$$

with the polar coordinate functions r and ϕ (note the difference to φ !) defined by

$$1 \pm i\beta_0\alpha_\tau = re^{\pm i\phi}, \quad r = \sqrt{1 + \beta_0^2\alpha_\tau^2}, \quad \phi = \arctan(\beta_0\alpha_\tau). \quad (112)$$

The n -fold integration by parts removes a polynomial of order n from the expansion of the logarithm in Eq. (103).

One can see that the result is an asymptotic expansion as the residual term, i.e. the last term in Eq. (111) is of the formal order α_τ^n . However, the obtained result is not a series expansion in the original coupling determined in the Euclidean domain but a more complicated system of functions related to it. The system of functions is ordered and the asymptotic expansion is valid in the sense of Poincaré. The system of functions is obtained by using the expression for the running coupling in Eq. (102) in the Euclidean domain and continuing it into the complex plane and onto the cut. When the analytic structure of the initial function is known, asymptotic expansions which converge fast for the first few terms (as a representation in the form of Eq. (111)) are more useful for practical calculations than formal convergent series that require many terms for getting a reasonable accuracy.

The expansion in Eq. (111) can give a better accuracy (for some n and α_τ) than a direct expansion in α_s . Indeed, this expansion includes a partial resummation of the π^2 terms which

is a consequence of the analytic continuation [39]. Therefore, the expansion can be understood as being done in terms of quantities defined on the cut. This is evident because in some sense the procedure consists in calculating derivatives of the spectral density on the cut. Because the region near the real axis is important, the continuation causes a change of the effective expansion parameter $\alpha_\tau \rightarrow \alpha_\tau / \sqrt{1 + \beta_0^2 \alpha_\tau^2}$. The first term in the expansion shown in Eq. (111) is just the value for the spectral density expressed through the coupling in the Euclidean domain. It can be considered as a change of the scheme at large values of s .

With the concise expression for the moments at hand one can change the form of the residual term. The relation

$$\begin{aligned} (n-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi} \right)^n \int_{-\pi}^{\pi} \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi / \pi)^n} &= \\ = 2\pi e^{-\pi/\beta_0 \alpha_\tau} - (n-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi} \right)^n \left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty} \right) \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi / \pi)^n} \end{aligned} \quad (113)$$

valid for any n leads to a representation of the zeroth order moment in the form

$$\begin{aligned} m_{00} &= \frac{1}{\pi \beta_0} \left\{ \pi e^{-\pi/\beta_0 \alpha_\tau} + \phi + \sum_{j=1}^{n-1} (j-1)! \left(\frac{\beta_0 \alpha_\tau}{\pi r} \right)^j \sin(j\phi) \right. \\ &\quad \left. - \frac{(n-1)!}{2} \left(\frac{\beta_0 \alpha_\tau}{\pi} \right)^n \left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty} \right) \frac{e^{i\varphi} d\varphi}{(1 + i\beta_0 \alpha_\tau \varphi / \pi)^n} \right\}. \end{aligned} \quad (114)$$

Here an explicit nonperturbative term $e^{-\pi/\beta_0 \alpha_\tau}$ has appeared. Eq. (114) and Eq. (111) are formally different but actually identical. Therefore, the choice for the expansion (or representation) for the moment is a question of calculating the residual term. This quantitative consideration is, of course, only possible if one has a concise expression for the function from which the series is generated as given by Eq. (105) in our case. Any conclusions about the precision or the analytic structure of the sum of the series based on the terms of the series only without a specification of the residual term are rather useless as one can see from Eq. (113).

We stress that the moments in Eq. (105) are analytic functions of α_τ for small values of the coupling α_τ . This means that the non-analytic piece in Eq. (113) cancels the corresponding piece in the residual term. If the residual term is dropped, the analytic structure drastically changes depending on which representation, either Eq. (111) or (114), is used. This demonstrates the danger of reaching conclusions about power corrections emerging from the extrapolation of the running to the IR region. Mathematically, or from a theoretical point of view, a conclusion of

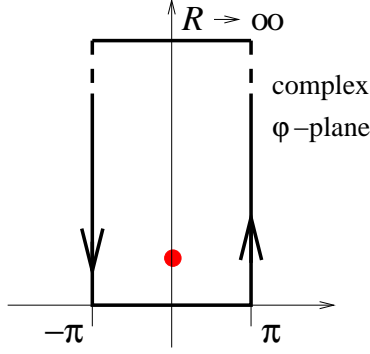


Figure 7: the integration contour in the complex φ -plane

this type is a hypothesis, especially having in mind that such power corrections are not necessary in any way for the consistency of the theory (in contrast to the OPE corrections which are well-defined in perturbation theory and are motivated by, say, explicitly distinguishing different channels). Physically, the use of such corrections is very difficult to appreciate if higher order terms of the expansion in α_s are taken into account, i.e. an explicit nonperturbative term in the expression, not being special in any general way, does not numerically differ from the higher order terms as well.

Yet another representation for the moments can be readily obtained. It is possible to recover the form of the moments as integrals over a spectral density (see below). But because moments are calculated as explicit functions, we can obtain a result by using a simple change of variables in a pure mathematical sense. In order to obtain this new representation we go to the complex plane in φ (see Fig. 7). As a technical trick we first integrate once by parts to obtain

$$m_{00} = \frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{(1 + e^{i\varphi})d\varphi}{1 + i\beta_0\alpha_\tau\varphi/\pi} \quad (115)$$

This representation is somewhat different from the representation in Eq. (111) for $n = 1$. They differ by an integral which can be explicitly computed,

$$\frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{d\varphi}{1 + i\beta_0\alpha_\tau\varphi/\pi} = \frac{1}{2\pi i\beta_0} \ln \left(\frac{1 + i\beta_0\alpha_\tau}{1 - i\beta_0\alpha_\tau} \right) = \frac{1}{\pi\beta_0} \arctan(\beta_0\alpha). \quad (116)$$

Now we consider the integration over a rectangular contour in the complex φ -plane. The part of the contour on the real axis from $-\pi$ to π leads to the moments. The integral over the

contour is given by the residue at the pole $\varphi = i\pi/\beta_0\alpha_\tau$. We thus have

$$\begin{aligned}
m_{00} &= \frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{(1 + e^{i\varphi})d\varphi}{1 + i\beta_0\alpha_\tau\varphi/\pi} = \\
&= \frac{i\alpha_\tau}{\pi} \left(\frac{1 + e^{i\varphi}}{i\beta_0\alpha_\tau/\pi} \right) \Big|_{\varphi=i\pi/\beta_0\alpha_\tau} + \\
&\quad - \frac{\alpha_\tau}{2\pi^2} \int_0^\infty \frac{(1 + e^{i\pi-\xi})id\xi}{1 + i\beta_0\alpha_\tau - \beta_0\alpha_\tau\xi/\pi} + \quad (\varphi = \pi + i\xi) \\
&\quad + \frac{\alpha_\tau}{2\pi^2} \int_0^\infty \frac{(1 + e^{-i\pi-\xi})id\xi}{1 - i\beta_0\alpha_\tau - \beta_0\alpha_\tau\xi/\pi} + \quad (\varphi = -\pi + i\xi) \\
&\quad + \frac{\alpha_\tau}{2\pi^2} \lim_{R \rightarrow \infty} \int_{-\pi}^{\pi} \frac{(1 + e^{i\tilde{\varphi}-R})d\tilde{\varphi}}{1 + i\beta_0\alpha_\tau\tilde{\varphi}/\pi - \beta_0\alpha_\tau R/\pi}. \quad (\varphi = \tilde{\varphi} + iR)
\end{aligned} \tag{117}$$

where we have indicated the substitutions for the different parts of the path. The last part vanishes because the integrand vanishes for $R \rightarrow \infty$ while the integration range is finite. We thus obtain

$$m_{00} = \frac{1}{\beta_0}(1 + e^{-\pi/\beta_0\alpha_\tau}) - \frac{1}{\beta_0} \int_0^\infty \frac{(1 - e^{-\xi})d\xi}{\pi^2 + (\xi - \pi/\beta_0\alpha_\tau)^2}. \tag{118}$$

Next we replace $-\pi/\beta_0\alpha_\tau = \ln(\Lambda^2/M_\tau^2)$ and substitute $-\xi = \ln(s/M_\tau^2)$, so that $e^{-\xi} = s/M_\tau^2$ and $-d\xi = ds/s$. One obtains

$$m_{00} = \frac{1}{\beta_0} \left(1 + \frac{\Lambda^2}{M_\tau^2} \right) - \frac{1}{\beta_0} \int_0^{M_\tau^2} \frac{(1 - s/M_\tau^2)ds}{(\pi^2 + \ln^2(s/\Lambda^2))s}. \tag{119}$$

A further transformation gives

$$m_{00} = \frac{1}{\beta_0} \left(\frac{\Lambda^2}{M_\tau^2} \right) + \frac{1}{\pi\beta_0} \int_0^{M_\tau^2} \arccos \left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}} \right) \frac{ds}{M_\tau^2}. \tag{120}$$

One easily recognizes this representation as an integration over the singularities of $\Pi(q^2)$ in Eq. (101). In addition to a cut along the positive semi-axis there appears also a part of the singularity on the negative real s -axis. This part is a pure mathematical feature of the concrete approximation chosen for $\Pi(q^2)$ and is not related to the physical content of the problem. Indeed, if moments are written as explicit functions we can calculate them in the way we find most convenient for a concrete application. The result reads

$$m_{00} = \int_{-\Lambda^2}^{M_\tau^2} \frac{\sigma(s)ds}{M_\tau^2} \tag{121}$$

with

$$\sigma(s) = \frac{1}{\beta_0} \theta(\Lambda^2 + s) \theta(-s) + \frac{1}{\pi\beta_0} \theta(s) \arccos \left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}} \right). \tag{122}$$

This formal result can be reformulated as integration over the spectrum $\sigma(s)$ using Cauchy's theorem. Indeed, from Eq. (101) one readily deduces the singularity of the resummed polarization function with the discontinuity

$$\text{Disc } \Pi(s) = \frac{2\pi i}{\beta_0} \left\{ \theta(\Lambda^2 + s)\theta(-s) + \frac{1}{\pi}\theta(s) \arccos \left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}} \right) \right\} \quad (123)$$

which coincides with $\sigma(s)$ in Eq. (122) and, thereby, simply represents the spectrum.

This spectral function $\sigma(s)$ is shown in Fig. 8. The part of the spectrum on the positive real axis is an analytic continuation of the renormalization group improved correlator function $\Pi(Q^2)$ to the cut [39, 55, 60, 61, 62]. It can be conveniently written in the form

$$\arccos \left(\frac{\ln(s/\Lambda^2)}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}} \right) = \arcsin \left(\frac{\pi}{\sqrt{\pi^2 + \ln^2(s/\Lambda^2)}} \right) = \arctan \left(\frac{\pi}{\ln(s/\Lambda^2)} \right) \quad (124)$$

with the appropriate branches of the relevant functions. Introducing

$$\alpha(s) = \frac{\pi}{\beta_0 \ln(s/\Lambda^2)} \quad (125)$$

one can write

$$\arctan \left(\frac{\pi}{\ln(s/\Lambda^2)} \right) = \arctan(\beta_0 \alpha(s)) \quad (126)$$

Taking the value of the spectral density at M_τ^2 as initial value,

$$\sigma(M_\tau^2) = \frac{1}{\pi\beta_0} \arctan(\beta_0 \alpha_\tau) = \frac{\alpha_\tau}{\pi} + O(\alpha^2). \quad (127)$$

we can introduce the continuum part of the spectral density,

$$\sigma_c(s) = \frac{1}{\pi\beta_0} \arctan(\beta_0 \alpha(s)). \quad (128)$$

The differential equation determining the continuum part $\sigma_c(s)$ through its initial value $\sigma(M_\tau^2)$ can be constructed by differentiating Eq. (128) with respect to s ,

$$s \frac{d\sigma_c(s)}{ds} = -\beta_0 \left(\frac{\alpha(s)}{\pi} \right)^2 \frac{1}{1 + \beta_0^2 \alpha(s)^2} \quad (129)$$

with $\alpha(s)$ taken from Eq. (125). By inverting Eq. (128) we have

$$\beta_0 \alpha(s) = \tan(\pi \beta_0 \sigma_c(s)) \quad \text{for } s > 0 \quad (130)$$

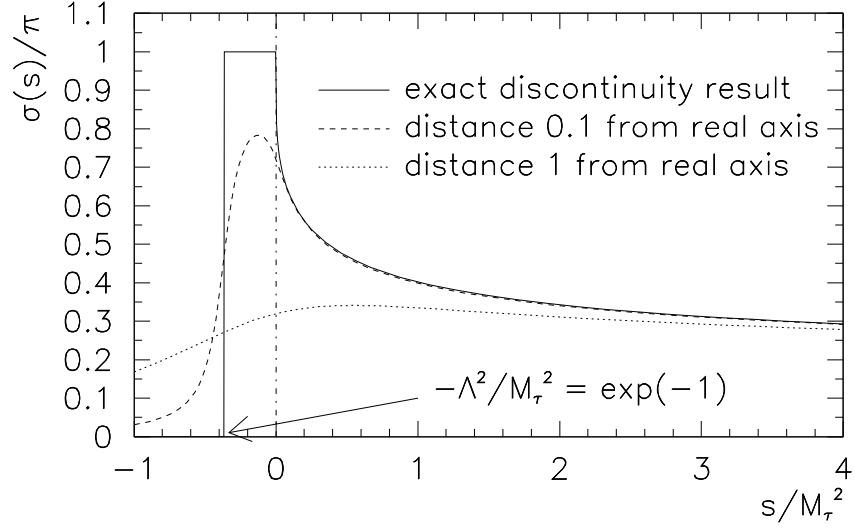


Figure 8: The spectral function $\sigma(s)$ as a function of s . Shown is the exact result for the discontinuity (full line) as well as results for non-closed circle integrals approaching the axis along a full circle from both sides, see Fig. 2. The numerical values for the distances orthogonal to the real axis towards the positive and negative imaginary semi-plane in s are given in units of M_τ^2 .

and therefore obtain

$$s \frac{d}{ds} \sigma_c(s) = -\frac{1}{\pi^2 \beta_0} \sin^2(\pi \beta_0 \sigma_c(s)) \quad \text{for } s > 0. \quad (131)$$

This equation can indeed be considered as an evolution equation for the spectral density $\sigma_c(s)$ determining $\sigma_c(s)$ through its initial value $\sigma(M_\tau^2)$. Therefore, one can simply introduce an effective charge

$$\alpha_M(s) = \pi \sigma_c(s) \quad \text{or} \quad a_M(s) = \frac{\alpha_M(s)}{\pi} = \sigma_c(s) \quad (132)$$

(see Eq. (16)). The evolution equation for $a_M(s)$ is then given by

$$s \frac{da_M(s)}{ds} = -\frac{1}{\pi^2 \beta_0} \sin^2(\pi \beta_0 a_M(s)). \quad (133)$$

Therefore, we now define the coupling as the value of the spectral density on the cut far from the IR region. This is a perturbation theory definition. The evolution of this coupling, however, is calculated by taking into account the analytic continuation. Then it has an IR fixed point with the coupling value

$$a_M(0) = 1/\beta_0 \quad (134)$$

(we assume that all transformations are made for small a_M so that this is the first IR point).

Starting from finite-order perturbation theory one therefore can find the all-order result because of analytic continuation and running in the IR domain – it happens to be finite (no poles for the spectrum at small values of s). Therefore, the function $\sigma_c(s)$ can be used to generate moments by integration on the cut. Note that in this approach the part of the spectrum at negative s is lost and cannot be recovered from finite-order perturbation theory.

Note, however, that the value at zero mentioned in Eq. (134) is not a universal quantity. If Adler's function starts with a proper power of the coupling constant as it is the case for gluonic observables, for instance, this picture will change. For instance, we take

$$D(Q^2) = \left(\frac{\alpha_E(Q^2)}{\pi} \right)^2. \quad (135)$$

Then the corresponding spectral density in leading order β -function approximation reads

$$\rho(s) = \frac{1}{\beta_0^2} \frac{1}{\ln^2(s/\Lambda^2) + \pi^2} = \frac{\alpha^2(s)}{\pi^2(1 + \beta_0^2 \alpha^2(s))}. \quad (136)$$

As in previous cases, we can define an effective coupling in the Euclidean domain,

$$\bar{a}_M(s) = \frac{\alpha(s)}{\pi \sqrt{1 + \beta_0^2 \alpha^2(s)}}. \quad (137)$$

For this effective coupling we indeed have (cf Eq. (134))

$$\bar{a}_M(0) = 0. \quad (138)$$

The β -function for the effective coupling obtained from Eq. (137),

$$\bar{\beta}_M(\bar{a}_M) = -\beta_0 \bar{a}_M^2 \sqrt{1 - (\pi \beta_0 \bar{a}_M)^2} = -\beta_0 \bar{a}_M^2 \left(1 - \frac{1}{2} (\pi \beta_0 \bar{a}_M)^2 + O(\bar{a}_M^4) \right) \quad (139)$$

differs from the β -function of the effective coupling a_M (c.f. Eq. (133))

$$\beta_M(a_M) = -\frac{1}{\pi^2 \beta_0} \sin(\pi \beta_0 a_M) = -\beta_0 a_M^2 \left(1 - \frac{1}{3} (\pi \beta_0 a_M)^2 + O(a_M^4) \right) \quad (140)$$

at next-to-leading order.

The consideration can be generalized to higher powers of the coupling in the Euclidean domain. For

$$D_p(Q^2) = \left(\frac{\alpha_E(Q^2)}{\pi} \right)^{p+1} \quad (141)$$

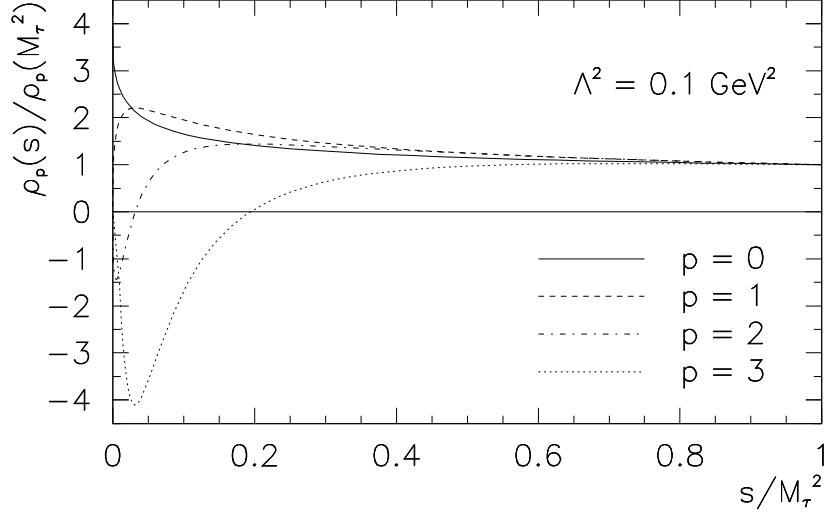


Figure 9: The spectral function $\rho_p(s)$ as a function of s , given by Eq. (142). All spectral functions for $p > 0$ are going to zero for $s \rightarrow 0$, for $p \geq 2$ they have nontrivial zeros and a fluctuating behaviour near the origin.

we obtain

$$\rho_p(s) = \frac{1}{\pi\beta_0} \frac{\sin(p\phi(s))}{pr^p(s)} \quad (142)$$

where

$$r(s) = \beta_0 \sqrt{\ln^2(s/\Lambda^2) + \pi^2}, \quad \tan \phi(s) = \frac{\pi}{\ln(s/\Lambda^2)} = \beta_0 \alpha(s). \quad (143)$$

For $p = 0$ we retain Eq. (128),

$$\rho_0(s) = \frac{1}{\pi\beta_0} \phi(s) = \frac{1}{\pi\beta_0} \arctan(\beta_0 \alpha(s)). \quad (144)$$

But for large values of p the function $\rho_p(s)$ starts to fluctuate when starting at $\rho(M_\tau^2)$ and approaching $s = 0$ where it finally reaches zero. Because if this behaviour, shown in Fig. 9 in some detail for the first four values of p , $\rho_p(s)$ fails to be interpreted as a (positive) spectral density in a region which enlarges with increasing values of p .

We conclude that the two recipes of resummation (on the contour and on the positive semi-axis for s) are different: they differ by the integral over the negative real semi-axis for s or, more generally, by the integral passing through the infrared region. This point is worth discussing in more detail. In the contour formulation it is not essential what particular point-by-point behaviour in the IR region exists. For the analytically continued correlator this is not

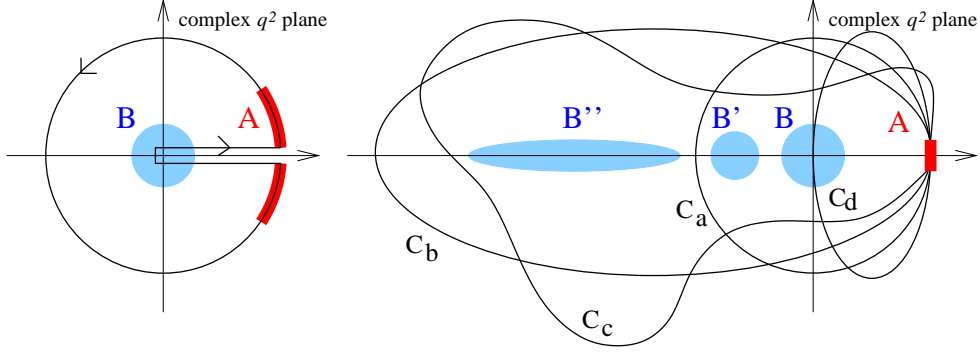


Figure 10: Contours in the complex plane with a fixation in the region A, taking into account possible occurrences of singularities. The figure part on the left hand side shows the standard circle path which circumvents the singular region B while the occurrence of other singularity regions as discussed in the text (regions B' and B'') may lead to different possibilities for choosing a path (C_a , C_b , C_c). The path C_d crosses the singularity region and, therefore, cannot be used from the perturbation theory point of view.

important unless the contour crosses a nonanalytic region. Whatever singularities exist in the IR region (Regions B, B', or B'' in Fig. 10), the contour includes them. This is the definition of resummation on the contour which is explicitly perturbative.

For the formulation on the cut the extrapolation of the running of the coupling constant to the IR region is crucial because it provides an interpretation of the spectral density in the IR region. One can formally write

$$\int_0^{M_\tau^2} \alpha(s) ds \quad (145)$$

and try to interpret this integral. The naive substitution of the expression in Eq. (125) leads to a singularity which is known as Landau pole. Still, further formal manipulations readily give ($t = \beta_0 \alpha(M_\tau^2) \ln(M_\tau^2/s)$)

$$\int_0^{M_\tau^2} \alpha(s) ds = \int_0^{M_\tau^2} \frac{\alpha(M_\tau^2) ds}{1 + \beta_0 \alpha(M_\tau^2) \ln(s/M_\tau^2)} = \frac{M_\tau^2}{\beta_0} \int_0^\infty \frac{e^{-t/\beta_0 \alpha(M_\tau^2)} dt}{1 - t}. \quad (146)$$

Both equations are not well-defined from the beginning, the second form being a Borel representation. The same problem can be reformulated as a divergence of the asymptotic series. Indeed, by expanding the expression for the running coupling under the integration sign in a

perturbation theory series one has

$$\int_0^{M_\tau^2} \alpha(s) ds = \sum_n n! \left(\frac{\beta_0 \alpha(M_\tau^2)}{\pi} \right)^n. \quad (147)$$

The summation of the series in Eq. (147) is equivalent to the interpretation of the integral, independent on what form is chosen for it, so either Eq. (145) or Eq. (146). Therefore, an integrable behaviour of the coupling constant at small s (whatever the definition of this quantity is) can solve the problem of divergence of the integral or the summation of the asymptotic series. But this solution is strongly model dependent because the extrapolation of the evolution into the IR region is essentially arbitrary and higher order corrections are important. The explicit form of the extrapolation in Eq. (122) gives an extrapolation motivated by analytic continuation. It can also be considered as a special change of the renormalization scheme [26]. Indeed, for the coupling a_M with the evolution given in Eq. (133) one obtains an IR fixed point. In any case the result depends on the IR region which is strongly nonperturbative.

As it has become popular as being used as an extrapolation, for the analytic continuation the moments are defined simply by

$$m_{00}^{\text{anal}} = \int_0^{M_\tau^2} \frac{\sigma_c(s) ds}{M_\tau^2}, \quad (148)$$

excluding totally the negative part of the spectrum that emerges in the exact treatment of quantities originally defined in the Euclidean domain (see Eq. (121)). Using Eq. (120) one can relate these moments to the moments on the contour by

$$m_{00}^{\text{anal}} = -\frac{1}{\beta_0} e^{-\pi/\beta_0 \alpha_\tau} + m_{00}. \quad (149)$$

The contour definition for the moments is a strictly perturbative one and in this particular case simply gives the analytic function of α_τ at the origin $\alpha_\tau = 0$, i.e. $m_{00}(\alpha_\tau)$ is analytic at small α_τ . Then the decomposition in Eq. (149) is at least meaningful, the right hand side does contain terms of different analytic structure. We can, of course, reproduce the contour moments by summing on the cut (which gives m_{00}^{anal}) and adding a pole (giving the “nonperturbative” term in Eq. (149)) which is an allowed extrapolation as well (because the IR region behaviour cannot be found in perturbation theory). But it is difficult to imagine that one could suggest such a treatment of the situation just as it is, if not derived naturally from the contour definition. Note

that this is completely analogous to the situation with Coulomb resummation: by summing on the cut $4m^2 < s < \infty$ one can never restore the pole contributions generated by the bound states. Only an explicit analytic continuation can reveal the whole analytic structure of the correlator. The result of Eq. (149) can readily be generalized to any power of α_τ as a basic quantity which is important for correlators of hadronic currents with nonvanishing anomalous dimensions.

Note that the problem of a particular choice for the extrapolation of the evolution into the IR region cannot be uniquely solved from the point of view of the perturbation theory because the IR region is nonperturbative and there is no tool to choose a “correct” extrapolation in cases when there is no explicit analytic structure known. Therefore, the extrapolation is rather arbitrary for basically all cases except the case of two-point correlators. Still, a particular choice for the extrapolation can be important numerically for the theoretical description of the available high precision data. In τ decays the special role of contour moments is related to perturbation theory – they are really computable in perturbation theory without any additional hypotheses. For the s^l -moments the problem is less important numerically while for the $(M_\tau^2 - s)^k$ -moments it is crucial (see Table 5). This is obvious because the $(M_\tau^2 - s)^k$ -moments are saturated at small s and the particular way of treating this region is important. In order to discuss this issue, we now leave the special case $k = l = 0$ and turn to the description of modified moments, discussing the differences.

6.3 The moments m_{0l}

In general, for any given moment m_{0l} one finds

$$m_{0l} = \frac{1}{\pi\beta_0} \left\{ \phi + \sum_{j=1}^{n-1} (j-1)! \left(\frac{\beta_0\alpha_\tau}{(l+1)\pi r} \right)^j \sin(j\phi) + \right. \\ \left. -(l+1)(-1)^{l+1} \frac{(n-1)!}{2} \left(\frac{\beta_0\alpha}{(l+1)\pi} \right)^n \int_{-\pi}^{\pi} \frac{e^{i(l+1)\varphi} d\varphi}{(1 + i\beta_0\alpha_\tau\varphi/\pi)^n} \right\}. \quad (150)$$

The representation obtained by using integration by parts shows an improvement in the convergence for moments at large l equivalent to the replacement $\alpha_\tau \rightarrow \alpha_\tau/(l+1)$. This coincides with conclusions drawn from finite-order perturbation theory. For arbitrary l one can also find

a representation containing an explicit nonperturbative term,

$$m_{0l} = \frac{1}{\pi\beta_0} \left\{ -\pi(-1)^{l+1}e^{-(l+1)\pi/\beta_0\alpha_\tau} + \phi + \sum_{j=1}^{n-1} (j-1)! \left(\frac{\beta_0\alpha_\tau}{(l+1)\pi r} \right)^j \sin(j\phi) \right. \\ \left. + (l+1)(-1)^{l+1} \frac{(n-1)!}{2} \left(\frac{\beta_0\alpha}{(l+1)\pi} \right)^n \left(\int_{-\pi}^{-\pi} + \int_{\pi}^{\infty} \right) \frac{e^{i(l+1)\varphi} d\varphi}{(1+i\beta_0\alpha_\tau\varphi/\pi)^n} \right\}. \quad (151)$$

These formulae give a direct generalization to large l moments and can be treated perturbatively.

6.4 The moments m_{k0}

The moments for $l = 0$ corresponding to the α_τ dependent part of the correlator in the leading order are given by

$$m_{k0} = -\frac{k+1}{2\pi\beta_0} \int_{-\pi}^{\pi} (1+e^{i\varphi})^k \ln(1+i\beta_0\alpha_\tau\varphi/\pi) e^{i\varphi} d\varphi. \quad (152)$$

These moments can also be expanded in a convergent series in α_τ for small values of α_τ , recovering the results from finite-order perturbation theory.

The usual arguments given for the preferred use of these moments from the theoretical point of view is that they suppress contributions close to the real axis where OPE is not applicable (region A in Fig. 10)). They do suppress these contributions because of $e^{i\varphi} \sim -1$ in this region. One can also obtain this result by looking at the integral measure for the moments m_{k0} ,

$$(1+e^{i\varphi})^k e^{i\varphi} d\varphi = (2\cos(\varphi/2))^k e^{ik\varphi/2} e^{i\varphi} d\varphi \quad (153)$$

where the suppression is given by the factor $(1+e^{i\varphi})^k$. The suppression of oscillations starts at larger values of the integration parameter φ than for the moments m_{0l} but there is a suppression by the power of $\cos(\varphi/2)$, $\cos(\varphi/2) = 0$ at $\varphi = \pm\pi$. However, one can see that these moments are almost insensitive to the value of α_τ for large k (or, more precisely, to the perturbative structure). It is clear that these functions determine integrals which really have no relation to perturbative running because such a running is completely suppressed. This makes an asymptotic expansion in α_τ almost useless (we could not find any efficient asymptotic expansion in α_τ for this moments at large values of k). However, it is straightforward to write down a nonperturbative representation for these moments at large values of k . A representation in terms of integral over the spectrum is particularly convenient. One has

$$m_{k0} = -\frac{k+1}{2\pi\beta_0} \int_{-\pi}^{\pi} (1+e^{i\varphi})^k \ln(1+i\beta_0\alpha_\tau\varphi/\pi) e^{i\varphi} d\varphi = \int_{-\Lambda^2}^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^k \frac{\sigma(s) ds}{M_\tau^2}. \quad (154)$$

$l = 3$	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.002	-0.003
$l = 2$	+0.000	+0.002	+0.004	+0.008	+0.013	+0.020	+0.030	+0.042
$l = 1$	-0.003	-0.009	-0.018	-0.032	-0.052	-0.078	-0.113	-0.160
$l = 0$	+0.186	+0.305	+0.396	+0.469	+0.530	+0.582	+0.626	+0.665
	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$

Table 5: Contribution of the integral taken over the interval from $-\Lambda^2$ to $+\Lambda^2$ for the moments m_{kl} in Eq. (159) relative to the integral taken over to the whole integration range.

The part of the spectrum on the negative real axis is given by a constant. For large values of k the integral is saturated at the left boundary of the integration region and reads

$$m_{k0} = \frac{1}{\beta_0} \left(\left(1 + \frac{\Lambda^2}{M_\tau^2} \right)^{k+1} - 1 \right) + (k+1) \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^k \frac{\sigma_c(s) ds}{M_\tau^2}. \quad (155)$$

The contribution of the integral is suppressed because the spectrum $\sigma_c(s)$ is a smooth and finite function leading to a finite result for the integral.

Another way to look at the problem is given by integrating by parts once. One has

$$m_{k0} = \frac{\alpha_\tau}{2\pi^2} \int_{-\pi}^{\pi} \frac{(1 + e^{i\varphi})^{k+1} d\varphi}{1 + i\beta_0 \alpha_\tau \varphi / \pi}. \quad (156)$$

In this case the boundary terms vanish. One obtains

$$m_{k0} = \frac{1}{\beta_0} \left(1 + \frac{\Lambda^2}{M_\tau^2} \right)^{k+1} - \frac{1}{\beta_0} \int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^{k+1} \frac{1}{\pi^2 + \ln^2(s/\Lambda^2)} \frac{ds}{s}. \quad (157)$$

Here it is easy to estimate the contribution of the integral as well. One has

$$\int_0^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^{k+1} \frac{1}{\pi^2 + \ln^2(s/\Lambda^2)} \frac{ds}{s} < \int_0^{M_\tau^2} \frac{1}{\pi^2 + \ln^2(s/\Lambda^2)} \frac{ds}{s} < 1. \quad (158)$$

Note that the whole function m_{k0} is still represented by a convergent series for small values of α_τ .

In general we determine the contribution of the nonperturbative region determined by the scale Λ^2 to the modified moments

$$m_{kl} = \int_{-\Lambda^2}^{M_\tau^2} \left(1 - \frac{s}{M_\tau^2} \right)^k \left(\frac{s}{M_\tau^2} \right)^l \frac{\sigma(s) ds}{M_\tau^2}. \quad (159)$$

$l = 3$	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.004
$l = 2$	+0.000	+0.001	+0.003	+0.005	+0.009	+0.014	+0.022	+0.031
$l = 1$	-0.006	-0.018	-0.036	-0.061	-0.093	-0.134	-0.187	-0.256
$l = 0$	+0.117	+0.195	+0.256	+0.308	+0.353	+0.392	+0.428	+0.460
	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$

Table 6: Contribution of the integral taken over the interval from $-\Lambda^2$ to 0 for the moments m_{kl} in Eq. (159) relative to the integral taken over to the whole integration range.

in order to consider the enhancement of this region in relation to the moment as a whole. In Table 5 we give values for the contribution of the interval range $[-\Lambda^2, \Lambda^2]$ to the contribution for the whole range $[-\Lambda^2, M_\tau^2]$ for different values of k and l , taking $\Lambda^2 = 0.2 \text{ GeV}^2$. A strong enhancement can be observed especially for the case $l = 0$ and large values for k . One observes that the first term is larger than the second one already for $k > 3$. Though weakened, the same tendency can be seen in Table 6 where we considered the contribution of the interval range $[-\Lambda^2, 0]$ relative to the whole range.

This observation also suggests the question of sensitivity of the moments to the parameters describing the strength of strong interactions, namely the values of a_M^τ and Λ . As expected, the s^l -moments are sensitive to the spectrum at large energies, i.e. $\sigma(M_\tau^2)$ which is almost equal to a_M^τ . The large k moments are directly sensitive to Λ as one can see from Eq. (157). However, perturbation theory is insufficient for these moments, and thus the theoretical formulae for large k moments like Eq. (157) cannot be confronted with experiment and should be supplemented by a quantitative hadronization mechanism.

6.5 Higher orders of running

The formulae obtained for the polarization function can readily be generalized to higher orders of the β -function. For Adler's function one has

$$D(Q^2) = 1 + a_E(Q^2). \quad (160)$$

The polarization function is then determined by the equation

$$-Q^2 \frac{d}{dQ^2} \Pi(Q^2) = 1 + a_E(Q^2). \quad (161)$$

A simple way to calculate $\Pi(Q^2)$ is to use the ansatz

$$\Pi(Q^2) = \ln \left(\frac{\mu^2}{Q^2} \right) + \ln f(a_E(Q^2)). \quad (162)$$

One then obtains a differential equation for $f(a_E)$ which reads

$$-Q^2 \frac{d}{dQ^2} \Pi(Q^2) = 1 - \frac{1}{f(a_E(Q^2))} Q^2 \frac{d}{dQ^2} f(a_E(Q^2)) = 1 - \frac{f'(a_E(Q^2))}{f(a_E(Q^2))} Q^2 \frac{da_E(Q^2)}{dQ^2}. \quad (163)$$

Using the renormalization group equation

$$Q^2 \frac{da_E(Q^2)}{dQ^2} = \beta(a_E(Q^2)) \quad (164)$$

one finds

$$f'(a_E(Q^2)) \beta(a_E(Q^2)) = -f(a_E(Q^2)) a_E(Q^2). \quad (165)$$

This differential equation can be solved in quadrature,

$$\ln f = - \int \frac{a_E da_E}{\beta(a_E)}. \quad (166)$$

The solution for $\Pi(Q^2)$ is therefore given by the trajectory

$$\Pi(Q^2) = - \int \frac{dQ^2}{Q^2} D(Q^2) = \ln \left(\frac{\mu^2}{Q^2} \right) - \int \frac{a_E da_E}{\beta(a_E)}. \quad (167)$$

Eq. (167) can be used for any β -function (with the restriction that no singularity should occur).

For $\beta(a) = -\beta_0 a^2$ we obtain the former result,

$$\beta(a) = -\beta_0 a^2 \quad \Rightarrow \quad \ln f = \frac{1}{\beta_0} \ln(a_E(Q^2)). \quad (168)$$

For $\beta(a) = -\beta_0 a^2 - \beta_1 a^3$ we explicitly obtain

$$\Pi(Q^2) = \ln \left(\frac{\mu^2}{Q^2} \right) + \frac{1}{\beta_0} \ln \left(\frac{\beta_0 a_E(Q^2)}{\beta_0 + \beta_1 a_E(Q^2)} \right). \quad (169)$$

The generalization to any polynomial function that may occur in perturbation theory is straightforward.

As a last note we comment on the Landau singularity which occurs at the leading order of the running. For two-point correlators the problem is completely solved by the contour integration to any order of the β -function. We stress that the pole is inside the circle (see the discussion in Ref. [57]). One cannot deform the contour to be close to the origin because this region is completely nonperturbative and should therefore be avoided. This situation is completely analogous to the situation with Coulombic poles [63] or with a spectrum below threshold for a gluonic operator induced by a heavy quark loop [64]. Any type of singularity is avoided by moving the contour far from the origin and keeping the singularity inside, thereby including also the integral of the spectrum.

7 Conclusion

We have shown that the direct s^l -moments are perturbative and can be reliably calculated within operator product expansion (OPE). The mixed moments with the weight functions $(1 - s/M_\tau^2)^k$ are strongly nonperturbative. This is natural and is expected in QCD with asymptotic freedom. We have analyzed finite-order perturbation theory, power corrections and resummation. The results agree with each other and the picture is completely consistent. An asymptotic analysis (large values for l or k) definitely disfavors the large k moments despite arguments that they suppress the non-OPE region along the contour – these arguments are not valid. In practical applications to τ decays and the determination of m_s the second mixed moment with weight function $(1 - s/M_\tau^2)^2$ is sometimes used. Along with the initial phase space weight $(1 - s/M_\tau^2)^2$ (see Eq. (8)) the total results give a moment with the weight $(1 - s/M_\tau^2)^4$ which is marginal in the sense that it enhances nonperturbative contributions. Our consideration shows that this moment is really on the border of what is still theoretically computable. The theoretical error becomes larger than one would expect for such an analysis. Note that an estimate of the systematic theoretical error, i.e. the error due to the functional dependence like perturbation series truncation, is complicated as compared to a simple statistical error of theoretical formulae due to uncertainties in the parameters used. The theoretical error for the moment with weight $(1 - s/M_\tau^2)^4$ appearing in the phenomenological analysis is big enough to be concerned. The point is whether we should worry about the size of this error in compar-

ison with the gain due to the improvement in the accuracy of the experimental moments or to trade this moment for a more reliable theoretical expression confronted with a less precise experimental moment. This still remains to be seen.

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